Chapter ② Problems

Final Exam Study Guide Solutions.

① $2c+4=4c-1 \Rightarrow 5=2c \Rightarrow c=\boxed{\frac{5}{2}}$

② (a). $\lim_{x\to 4}\frac{x^2-16}{x-4}=\lim_{x\to 4}\frac{\cancel{(x-4)}(x+4)}{\cancel{x-4}}=\lim_{x\to 4}(x+4)=\boxed{8}$

(b). $\lim_{x\to 4}\frac{\sqrt{x}-2}{3(x-4)}=\lim_{x\to 4}\frac{(\sqrt{x}-2)(\sqrt{x}+2)}{3(x-4)(\sqrt{x}+2)}=\lim_{x\to 4}\frac{\cancel{x-4}}{3\cancel{(x-4)}(\sqrt{x}+2)}=$

$=\lim_{x\to 4}\frac{1}{3(\sqrt{x}+2)}=\boxed{\frac{1}{12}}$

(c). $\lim_{x\to 0}\frac{\sin(8x)}{x}=\lim_{x\to 0}\frac{\sin(8x)}{8x}\cdot 8=\boxed{8}$

(d). $\lim_{x\to 0+}\frac{\sin(5x)}{|x|}=\lim_{x\to 0+}\frac{\sin(5x)}{x}=\boxed{5}$

(e). $\lim_{x\to 0-}\frac{\sin(3x)}{|x|}=\lim_{x\to 0-}\frac{\sin(3x)}{-x}=\boxed{-3}$

(f). $\lim_{x\to 3}\frac{x-3}{x^3-9x}=\lim_{x\to 3}\frac{\cancel{x-3}}{x\cancel{(x-3)}(x+3)}=\frac{1}{3\cdot 6}=\boxed{\frac{1}{18}}$

(g). $\lim_{x\to 0}\frac{\sqrt{5+x}-\sqrt{5-x}}{x}=\lim_{x\to 0}\frac{(\sqrt{5+x}-\sqrt{5-x})(\sqrt{5+x}+\sqrt{5-x})}{x(\sqrt{5+x}+\sqrt{5-x})}=$

$=\lim_{x\to 0}\frac{2\cancel{x}}{\cancel{x}(\sqrt{5+x}+\sqrt{5-x})}=\frac{2}{2\sqrt{5}}=\boxed{\frac{1}{\sqrt{5}}}$

(h). $\lim_{x\to 0}\frac{\tan(4x)}{\tan(8x)}=\lim_{x\to 0}\frac{\frac{\tan(4x)}{4x}\cdot 4\cancel{x}}{\frac{\tan(8x)}{8x}\cdot 8\cancel{x}}=\frac{4}{8}=\boxed{\frac{1}{2}}$

③ (a). $\lim_{x\to 1-}f(x)=\boxed{3}$ (c). $\lim_{x\to 2-}f(x)=\boxed{\infty}$ (e). $\lim_{x\to 4-}f(x)=\boxed{6}$

(b). $\lim_{x\to 1+}f(x)=\boxed{3}$ (d). $\lim_{x\to 2+}f(x)=\boxed{2}$ (f). $\lim_{x\to 4+}f(x)=\boxed{3}$

$x=1$: Removable Discontinuity

$x=2$: Infinite Discontinuity

$x=4$: Jump Discontinuity

Chapter ③ Problems

① $x^2+3=4x \Rightarrow x^2-4x+3=0 \Rightarrow (x-3)(x-1)=0 \Rightarrow \boxed{x=1,3}$

② $p(1)=1+a+b=4$

$p'(x)=2x+a \Rightarrow p'(0)=a \Rightarrow \boxed{a=1}$

$1+a+b=4 \Rightarrow 1+1+b=4 \Rightarrow \boxed{b=2}$

③ $$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\left(-5x^2+6x+3hx-7h+12h^2\right)=$$

$$=\boxed{-5x^2+6x}$$

④ $$h'(0)=\frac{f'(0)g(0)-f(0)g'(0)}{g^2(0)}=\frac{(-2)(1)-(3)(5)}{1^2}=\boxed{-17}$$

⑤ $$f'(x)=\frac{e^x(x^2+1)-e^x\cdot 2x}{(x^2+1)^2}$$

$$f'(1)=\frac{e\cdot 2-e\cdot 2}{2^2}=\boxed{0}$$

⑥ $f'(x)=\frac{1}{2}x^{-1/2}$; $f''(x)=\frac{1}{2}\cdot\left(-\frac{1}{2}x^{-3/2}\right)=-\frac{1}{4}x^{-3/2} \Rightarrow \boxed{f''(1)=-\frac{1}{4}}$

⑦ $f'(x)=5e^x\cos(x)-5e^x\sin(x)=\boxed{5e^x(\cos(x)-\sin(x))}$.

⑧ $f'(x)=-4e^x-4xe^x=-4e^x(1+x)$. $\Rightarrow$ Slope : $f'(1)=-8e$

$f(1)=-4e \Rightarrow$ Point : $(1,-4e)$

Equation of Tangent Line:

$$y-(-4e)=-8e(x-1)$$

$$y+4e=-8ex+8e$$

$$\boxed{y=-8ex+4e}$$

⑨ $$f'(x)=\frac{5(x+4)-5x}{(x+4)^2}=\frac{20}{(x+4)^2} \Rightarrow f'(1)=\frac{20}{25}=\frac{4}{5}\ \text{(Slope)}$$

Equation of line: $y-1=\frac{4}{5}(x-1)$

$$y=\frac{4}{5}x-\frac{4}{5}+1 \Rightarrow \boxed{y=\frac{4}{5}x+\frac{1}{5}}$$

(10) $f'(x)=\frac{1}{2\sqrt{x}} \Rightarrow f'(3)=\frac{1}{2\sqrt{3}}$ (Slope)

Equation: $y-\sqrt{3}=\frac{1}{2\sqrt{3}}(x-3) \Rightarrow y-\sqrt{3}=\frac{1}{2\sqrt{3}}x-\frac{1}{2}\sqrt{3}$

$$\Rightarrow \boxed{y=\frac{1}{2\sqrt{3}}x-\frac{3}{2}\sqrt{3}}$$

(11) (a). $y=x^{5x}=[e^{\ln(x)}]^{5x}=e^{5x\cdot\ln(x)}$

$y'=e^{5x\cdot\ln(x)}(5x\cdot\ln(x))'=x^{5x}\left(5\ln(x)+5x\cdot\frac{1}{x}\right)=\boxed{x^{5x}(5\ln(x)+5)}$

(b). $y=x^{3^x}=[e^{\ln(x)}]^{3^x}=e^{3^x\cdot\ln(x)}$

$y'=e^{3^x\cdot\ln(x)}(3^x\cdot\ln(x))'=x^{3^x}\left(3^x\ln(3)\cdot\ln(x)-3^x\cdot\frac{1}{x}\right)=$

$$=\boxed{x^{3^x}\left(3^x\cdot\ln(3)\cdot\ln(x)-\frac{3^x}{x}\right)}$$

(12) (a). $y'=(2x+8x^3)\cos(x)-(x^2-1+2x^4)\sin(x)$.

(b). $y'=5e^x\sin(x)+5e^x\cos(x)=5e^x(\sin(x)+\cos(x))$.

(c). $y'=5x^4\cos(x)-x^5\sin(x)$

(d). $y'=\dfrac{5(\sin(x)+\cos(x))-5x(\cos(x)-\sin(x))}{[\sin(x)+\cos(x)]^2}$

(e). $y'=\dfrac{1}{1+\cos^2(4x)}(-\sin(4x)\cdot 4)=\dfrac{-4\sin(4x)}{1+\cos^2(4x)}$.

(f). $y'=2\arcsin(x)+\dfrac{2x}{\sqrt{1-x^2}}$

(g). $y'=6\cdot\dfrac{1}{\sqrt{1-(x^2)^2}}\cdot 2x=\dfrac{12x}{\sqrt{1-x^4}}$

(h). $y'=\dfrac{3+\frac{5}{2\sqrt{5x}}}{2\sqrt{3x+\sqrt{5x}}}$

(i). $y'=3(x+\sin(x))^2(1+\cos(x))$.

(j). $y'=e^{8-x^2}(-2x)=-2xe^{8-x^2}$

(k). $y' = -\sin(\sin(x)) \cdot \cos(x)$.

(l). $y' = 7e^{x\sin(x)} \, (\sin(x) + x\cos(x))$.

(m). $y' = 2\sin(\cos(x^3)) \cdot (-\sin(x^3) \cdot 3x^2) = -6x^2 \sin(\cos(x^3)) \cdot \sin(x^3)$.

(n). $y' = \dfrac{3}{\sin(x) + \cos(x)} \cdot (\cos(x) - \sin(x))$

(o). $y' = 5^{x^2 - x^4 + 3} \cdot \ln(5) \cdot (2x - 4x^3)$

(p). $y' = 4^{\sin(x^2)} \cdot \ln(4) \cdot \cos(x^2) \cdot 2x$

(q). $y' = \dfrac{\frac{1}{3x} \cdot 3 \cdot \sin(x) - \ln(3x) \cdot \cos(x)}{\sin^2(x)} = \dfrac{\frac{\sin(x)}{x} - \ln(3x) \cdot \cos(x)}{\sin^2(x)}$.

(r). $y' = \ln(x) + x \cdot \frac{1}{x} - 1 = \boxed{\ln(x)}$

(s). $y' = \dfrac{1}{\cos(\sin(x))} \cdot (-\sin(\sin(x)) \cdot \cos(x))$

Chapter ④ Problems

① $f'(x) = \ln(4x) + x \cdot \frac{1}{x} = \ln(4x) + 1$

$f'(x) = 0 \Rightarrow \ln(4x) = -1 \Rightarrow 4x = e^{-1} \Rightarrow x = \boxed{\frac{1}{4e}}$

② (a): $(-1, 1) \cup (5, 9)$
(b): $(1, 5)$
(c): $(-1, 0) \cup (3, 7)$
(d): $(0, 3) \cup (7, 9)$

③ $f'(x) = 4x - 4 \Rightarrow x = 1$ Critical Point

x	$f(x)$	
1	−2	← <u>min</u>
−1	6	← <u>max</u>
2	0	

$f(x) = 2x^2 - 4x$
$f(1) = 2 - 4 = -2$
$f(-1) = 2 + 4 = 6$
$f(2) = 8 - 8 = 0$

④ $f'(x) = e^{2x} + 2x e^{2x} = (1 + 2x) e^{2x}$
$f'(x) = 0 \Rightarrow 1 + 2x = 0 \Rightarrow x = \boxed{-\frac{1}{2}}$

⑤ $f'(x) = \frac{(x^{10}+6) - x \cdot 10x^9}{(x^{10}+6)^2} = \frac{x^{10} + 6 - 10x^{10}}{(x^{10}+6)^2} = \frac{6 - 9x^{10}}{(x^{10}+6)^2}$

$f'(x) = 0 \Rightarrow 6 - 9x^{10} = 0 \Rightarrow 9x^{10} = 6 \Rightarrow 3x^{10} = 2 \Rightarrow x^{10} = \frac{2}{3} \Rightarrow$
$\Rightarrow x = \boxed{\sqrt[10]{\frac{2}{3}}}$

⑥ $f'(x) = 3 - 15 \cdot \frac{1}{x} = 3 - \frac{15}{x} = \frac{3x - 15}{x}$

$f'(x) = 0 \Rightarrow 3x - 15 = 0 \Rightarrow x = \boxed{5}$ <u>Critical Point</u>

x	0+ 5
$f'(x)$	− − − 0 + + +

$\searrow \quad \nearrow$

f has a <u>local minimum</u> at $x = 5$

⑦ $f'(x) = (2x)e^x + (x^2+3)e^x = (x^2 + 2x + 3)e^x$
$f''(x) = (2x+2)e^x + (x^2 + 2x + 3)e^x = (x^2 + 4x + 5)e^x$
$f''(x) = 0 \Rightarrow x^2 + 4x + 5 = 0 \Rightarrow \quad x = \frac{-4 \pm \sqrt{16 - 20}}{2}$ ← <u>No Solutions</u>

L'Hôpital:

8 (a). $\lim\limits_{x\to 0}\frac{11^x-3^x}{x}\underset{\frac{0}{0}}{=}\lim\limits_{x\to 0}\frac{11^x\ln(11)-3^x\ln(3)}{1}=\boxed{\ln(11)-\ln(3)}$

(b). $\lim\limits_{x\to 0}\frac{x^2}{1-\cos(x)}\underset{\frac{0}{0}}{=}\lim\limits_{x\to 0}\frac{2x}{\sin(x)}\underset{\frac{0}{0}}{=}\lim\limits_{x\to 0}\frac{2}{\cos(x)}=\boxed{2}$

(c). $\lim\limits_{x\to 0}\frac{\sin(5x)}{\tan(3x)}\underset{\frac{0}{0}}{=}\lim\limits_{x\to 0}\frac{5\cos(5x)}{\frac{1}{\cos^2(3x)}\cdot 3}=\boxed{\frac{5}{3}}$

(d). $\lim\limits_{x\to 0}\frac{e^x-1}{\sin(4x)}\underset{\frac{0}{0}}{=}\lim\limits_{x\to 0}\frac{e^x}{4\cos(4x)}=\boxed{\frac{1}{4}}$

(e). $\lim\limits_{x\to 1}\frac{e-e^x}{\ln(x)}\underset{\frac{0}{0}}{=}\lim\limits_{x\to 1}\frac{-e^x}{\frac{1}{x}}=\boxed{-e}$

9 $\lim\limits_{x\to\infty}\left(\frac{14x}{14x+6}\right)^{4x}=\lim\limits_{x\to\infty}e^{\ln\left(\frac{14x}{14x+6}\right)^{4x}}=e^{\lim\limits_{x\to\infty}\ln\left(\frac{14x}{14x+6}\right)^{4x}}$

$=\boxed{e^{-\frac{12}{7}}}$

$\lim\limits_{x\to\infty}\ln\left(\frac{14x}{14x+6}\right)^{4x}=\lim\limits_{x\to\infty}4x\cdot\ln\left(\frac{14x}{14x+6}\right)=$

$=\lim\limits_{x\to\infty}\frac{4\cdot\ln\left(\frac{14x}{14x+6}\right)}{\frac{1}{x}}\underset{\frac{0}{0}}{=}\lim\limits_{x\to\infty}\frac{4\cdot\frac{14x+6}{14x}\cdot\frac{14(14x+6)-14x\cdot 14}{(14x+6)^2}}{-\frac{1}{x^2}}$

$=4\lim\limits_{x\to\infty}-x^{\cancel{2}}\cdot\cancel{\frac{14x+6}{14x}}\cdot\frac{\cancel{14}\cdot 6}{(14x+6)^{\cancel{2}}}=$

$=4\lim\limits_{x\to\infty}\left(\frac{-6x}{14x+6}\right)=4\cdot\frac{-6}{14}=4\cdot\frac{-3}{7}=\boxed{\frac{-12}{7}}$

(10) $f'(x) = \int (4x+3)\,dx = 2x^2 + 3x + C_1$

$\left.\begin{array}{l} f'(0) = 1 \\ f'(0) = C_1 \end{array}\right\} \Rightarrow C_1 = 1 \Rightarrow \boxed{f'(x) = 2x^2 + 3x + 1}$

$f(x) = \int (2x^2 + 3x + 1)\,dx = 2\frac{x^3}{3} + 3\frac{x^2}{2} + x + C_2$

$\left.\begin{array}{l} f(0) = 3 \\ f(0) = C_2 \end{array}\right\} \Rightarrow C_2 = 3 \Rightarrow \boxed{f(x) = \frac{2x^3}{3} + \frac{3x^2}{2} + x + 3}$

(11) $f(x) = \int (2e^x - 5)\,dx = 2e^x - 5x + C$

$\left.\begin{array}{l} f(0) = 2 + C \\ f(0) = 5 \end{array}\right\} \Rightarrow 2 + C = 5 \Rightarrow \boxed{C = 3}$

(12) $F'(x) = C_1 \sin(x) + C_1 x \cos(x) + C_2(-\sin(x))$

$= (C_1 - C_2)\sin(x) + C_1 x \cos(x)$

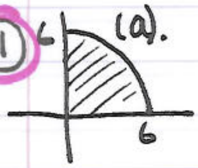
$= 4\sin(x) + 2x\cos(x)$

$C_1 - C_2 = 4$

$\boxed{C_1 = 2} \Rightarrow 2 - C_2 = 4 \Rightarrow \boxed{C_2 = -2}$

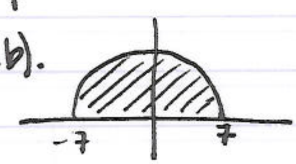
Chapter (5) Problems

(1) (a).



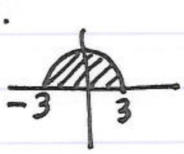
$\frac{1}{4} \cdot 36\pi = \boxed{9\pi}$

(b).



$\frac{1}{2} \cdot 49\pi = \boxed{\frac{49\pi}{2}}$

(c).



$\frac{1}{2} \cdot 9\pi = \boxed{\frac{9\pi}{2}}$

(2) (a). $\int_1^2 f(x)\,dx = \boxed{2}$ (b). $\int_4^2 f(x)\,dx = \boxed{-8}$

(3) $\int_0^1 \left(\frac{d}{dt}\sqrt{3+t^4}\right)dt = \sqrt{3+t^4}\Big|_0^1 = \sqrt{4}-\sqrt{3} = \boxed{2-\sqrt{3}}$

(4) $\int_0^1 \left(\frac{d}{dx}\ln\left(\frac{4x+1}{2x+4}\right)\right)dx = \ln\left(\frac{4x+1}{2x+4}\right)\Big|_0^1 = \boxed{\ln\left(\frac{5}{6}\right)-\ln\left(\frac{1}{4}\right)}$

(5) $f(x) = \int_x^{x^2} t\,dt = \frac{t^2}{2}\Big|_x^{x^2} = \frac{x^4-x^2}{2}$

$f'(x) = \frac{4x^3-2x}{2} = \boxed{2x^3-x}$

(6) $f(x) = \int_{\sqrt{x}}^{x} t^2\,dt = \frac{t^3}{3}\Big|_{\sqrt{x}}^{x} = \frac{x^3}{3} - \frac{(\sqrt{x})^3}{3}$

$f'(x) = \frac{3x^2}{3} - \frac{3(\sqrt{x})^2\cdot\frac{1}{2\sqrt{x}}}{3} = x^2 - \frac{x}{2\sqrt{x}} = \boxed{x^2-\frac{\sqrt{x}}{2}}$

(7) $f(x) = \int_4^{x^4}\sqrt{t^2+9}\,dt$

$A(x) = \int_4^{x}\sqrt{t^2+9}\,dt \Rightarrow$ by FTC II: $A'(x) = \sqrt{x^2+9}$

$f(x) = A(x^4) \Rightarrow f'(x) = A'(x^4)\cdot 4x^3 = \boxed{4x^3\sqrt{x^8+9}}$

(8) $f(x) = \int_1^{\sqrt{x}} \ln(t^2)\,dt$

$A(x) = \int_1^{x} \ln(t^2)\,dt \Rightarrow$ by FTC II: $A'(x) = \ln(x^2)$

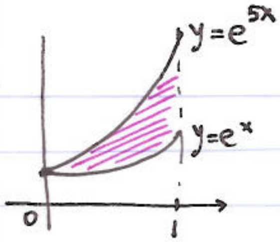
$f(x) = A(\sqrt{x}) \Rightarrow f'(x) = A'(\sqrt{x})\cdot\frac{1}{2\sqrt{x}} = \boxed{\frac{\ln(x)}{2\sqrt{x}}}$

(9) $\int_0^3 f(x)\,dx = \int_0^1 x^2\,dx + \int_1^3 x\,dx = \frac{x^3}{3}\Big|_0^1 + \frac{x^2}{2}\Big|_1^3 = \frac{1}{3}+\frac{9}{2}-\frac{1}{2} = \boxed{\frac{13}{3}}$

(10) $\int_0^5 f(x)\,dx = \int_0^2 x\,dx + \int_2^5 \frac{1}{x}\,dx = \frac{x^2}{2}\Big|_0^2 + \ln(x)\Big|_2^5 = \boxed{2+\ln(5)-\ln(2)}$

(11), (12) See #20, 21 in the Solutions to Study Guide for Exam 4.

Chapter ⑥ Problems

① 

$$A = \int_0^1 (e^{5x} - e^x)\,dx = \left(\frac{1}{5}e^{5x} - e^x\right)\Big|_0^1 =$$

$$= \frac{1}{5}e^5 - e - \left(\frac{1}{5} - 1\right) =$$

$$= \boxed{\frac{1}{5}e^5 - e + \frac{4}{5}}$$

② $V = \int_0^{12} (9\pi)\,dx = 9\pi x \Big|_0^{12} = 9\pi(12-0) = 9 \cdot 12\pi = \boxed{108\pi}$

③ $m = \frac{1}{2}\int_0^2 2xe^{x^2}\,dx = \frac{1}{2}\, e^{x^2}\Big|_0^2 = \boxed{\frac{1}{2}(e^4 - 1)}$

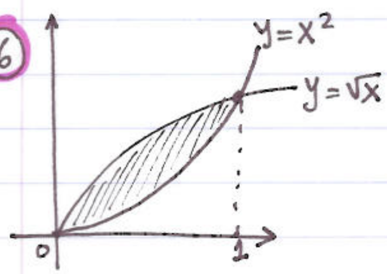
④ $V = \pi \int_0^3 (x^2 - 3x)^2\,dx = \pi \int_0^3 (x^4 - 6x^3 + 9x^2)\,dx = \pi\left(\frac{x^5}{5} - \frac{6x^4}{4} + 3x^3\right)\Big|_0^3 =$

$$= \pi\left(\frac{3^5}{5} - \frac{6 \cdot 3^4}{4} + 3 \cdot 3^3\right) = \pi \cdot \left(\frac{3^5}{5} - \frac{2 \cdot 3^5}{4} + 3^4\right) =$$

$$= \pi \cdot 3^4\left(\frac{3}{5} - \frac{6}{4} + 1\right) = \pi \cdot 3^4\left(\frac{3}{5} - \frac{3}{2} + 1\right) = \pi \cdot 81 \cdot \frac{1}{10}$$

$$= \boxed{8.1\pi}$$

⑤ $V = \pi\int_0^3 e^{2x}\,dx = \frac{\pi}{2} \cdot e^{2x}\Big|_0^3 = \boxed{\frac{\pi}{2}(e^6 - 1)}$

⑥ 

$$V = \pi\int_0^1 (\sqrt{x})^2 - (x^2)^2\,dx = \pi\int_0^1 (x - x^4)\,dx$$

$$= \pi \cdot \left(\frac{x^2}{2} - \frac{x^5}{5}\right)\Big|_0^1 = \pi \cdot \left(\frac{1}{2} - \frac{1}{5}\right) = \boxed{\frac{3\pi}{10}}$$

Chapter ⑧ Problems

① $y = 3x + 1 \Rightarrow y' = 3 \Rightarrow s = \int_0^9 \sqrt{1+9}\,dx = \boxed{9\sqrt{10}}$

② $y = 4x + 3 \Rightarrow y' = 4 \Rightarrow S = 2\pi\int_0^1 (4x+3)\sqrt{1+16}\,dx =$

$$= 2\pi\sqrt{17}\int_0^1 (4x+3)\,dx =$$

$$= 2\pi\sqrt{17} \cdot (2x^2 + 3x)\Big|_0^1 = \boxed{10\pi\sqrt{17}}$$