*

1. Find each of the integrals below:

a). [5 pts.] $\int \frac{x^3}{x^4+1}\,dx.$

Substitution: $u = x^4+1$
$du = 4x^3\,dx$
$\frac{1}{4}\,du = x^3\,dx$

$$\int \frac{x^3}{x^4+1}\,dx = \int \frac{1}{u}\,\frac{1}{4}\,du = \frac{1}{4}\ln|u| + C$$
$$= \boxed{\frac{1}{4}\ln(x^4+1) + C}$$

b). [5 pts.] $\int \frac{\ln(x)}{x^2}\,dx.$

By parts: $u = \ln x \qquad dv = \frac{1}{x^2}\,dx$
$du = \frac{1}{x}\,dx \qquad v = -\frac{1}{x}$

$$\int \frac{\ln x}{x^2}\,dx = \frac{-1}{x}\ln x - \int \frac{-1}{x^2}\,dx$$
$$= \boxed{\frac{-1}{x}\ln x - \frac{1}{x} + C}$$

*

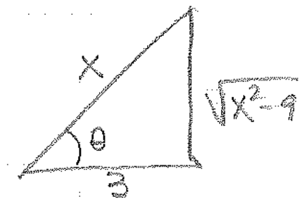
2. Find the integral

$$\int \frac{\sqrt{x^2-9}}{x}\,dx.$$

Trig Sub: $x = 3\sec\theta$, $\theta \in [0, \pi/2) \cup [\pi, 3\pi/2)$

$dx = 3\sec\theta\tan\theta\,d\theta$

$\sqrt{x^2-9} = 3\tan\theta$



$\sec\theta = \frac{x}{3}$

$\cos\theta = \frac{3}{x}$

$$\int \frac{\sqrt{x^2-9}}{x}\,dx = \int \frac{3\tan\theta}{\cancel{3\sec\theta}}\,\cancel{3\sec\theta}\tan\theta\,d\theta$$

$$= \int 3\tan^2\theta\,d\theta$$

$$= \int 3(\sec^2\theta - 1)\,d\theta$$

$$= \int (3\sec^2\theta - 3)\,d\theta$$

$$= 3\tan\theta - 3\theta + C$$

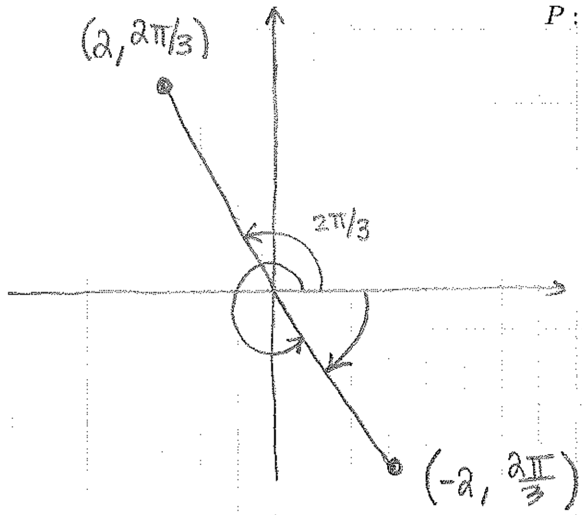
$$= 3\cdot\frac{\sqrt{x^2-9}}{3} - 3\sec^{-1}\left(\frac{x}{3}\right) + C$$

$$= \boxed{\sqrt{x^2-9} - 3\sec^{-1}\left(\frac{x}{3}\right) + C}$$

$$\text{or } \boxed{\sqrt{x^2-9} - 3\cos^{-1}\left(\frac{3}{x}\right) + C}$$

3. a). [3 pts.] Plot in the Cartesian plane the point P, given by polar coordinates:

$$P: \left(-2, \frac{2\pi}{3}\right).$$



$\frac{2\pi}{3} + \frac{6\pi}{3}$

b). [2 pts.] Find two different pairs of polar coordinates (r, θ) to describe the point P, one with $r > 0$ and one with $r < 0$.

Pair 1: $(r > 0)$ $\boxed{\left(2, -\pi/3\right); \left(2, \frac{5\pi}{3}\right)}$ Pair 2: $(r < 0)$ $\boxed{\left(-2, \frac{8\pi}{3}\right); \left(-2, \frac{-4\pi}{3}\right)}$

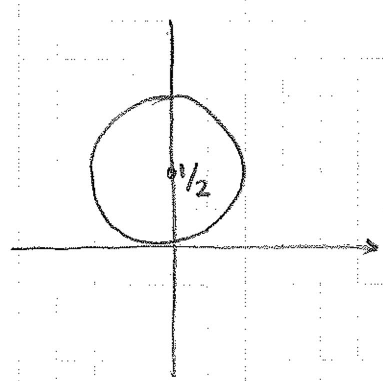
c). [5 pts.] Find the Cartesian equation of the polar curve:

$$r = \sin\theta,$$

and plot the curve.

$$r^2 = r\sin\theta$$
$$x^2 + y^2 = y$$
$$x^2 + y^2 - y = 0$$
$$x^2 + y^2 - y + \frac{1}{4} = \frac{1}{4}$$
$$\boxed{x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}}$$

Circle centered at $(0, 1/2)$ w/ radius $1/2$



*

4. Determine whether or not each series below converges or diverges. Justify your answer and state any Series Tests you use.

a). [4 pts.] $\sum_{n=1}^{\infty} \frac{\sqrt{n^5+n+2}}{\sqrt[4]{9n^{12}+7}}$

Dominating power: $\frac{n^{5/2}}{n^{12/4}} = \frac{n^{5/2}}{n^3} = \frac{1}{n^{1/2}}$

LCT with $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$, divergent p-series $(p = 1/2)$

$$\lim_{n\to\infty} \frac{\sqrt{n^5+n^2+2}\;\, n^{1/2}}{\sqrt[4]{9n^{12}+7}} = \lim_{n\to\infty} \frac{\sqrt{n^6+n^2+2}}{\sqrt[4]{9n^{12}+7}} = \frac{1}{\sqrt[4]{9}} \neq 0$$

$\Rightarrow$ by LCT, both series diverge.

b). [4 pts.] $\sum_{n=1}^{\infty} \frac{n}{2n^3+1}$

$$\frac{n}{2n^3+1} < \frac{n}{2n^3} = \frac{1}{2n^2}$$

$\sum_{n=1}^{\infty} \frac{1}{2n^2}$ converges (p-series w/ $p=2$)

$\Rightarrow$ S converges by Comparison Test.

c). [2 pts.] $\sum_{n=1}^{\infty} \arctan(n)$

$\lim_{n\to\infty} \arctan(n) = \frac{\pi}{2} \neq 0 \Rightarrow$ S diverges by Test for Divergence.

*

5. Find the integrals:

a). [5 pts.] $\int_{-1}^{0} \frac{x+1}{x^2+x-2}\,dx$

$$\frac{x+1}{x^2+x-2} = \frac{x+1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$$

$$x+1 = A(x+2) + B(x-1)$$

$$x=-2: \quad -1 = -3B \Rightarrow B = 1/3$$

$$x=1: \quad 2 = 3A \Rightarrow A = 2/3$$

$$\int_{-1}^{0} \frac{x+1}{x^2+x-2}\,dx = \int_{-1}^{0} \left(\frac{2/3}{x-1} + \frac{1/3}{x+2}\right)dx$$

$$= \left(\frac{2}{3}\ln|x-1| + \frac{1}{3}\ln|x+2|\right)\Big|_{-1}^{0}$$

$$= \left(\frac{2}{3}\ln(1) + \frac{1}{3}\ln(2)\right) - \left(\frac{2}{3}\ln(2) + \frac{1}{3}\ln(1)\right)$$

$$= \frac{1}{3}\ln(2) - \frac{2}{3}\ln(2) = \boxed{-\frac{1}{3}\ln(2)}$$

b). [5 pts.] $\int_{2}^{\infty} \frac{1}{(x-1)^{5/2}}\,dx$

$$\int \frac{1}{(x-1)^{5/2}}\,dx = \int u^{-5/2}\,du = \frac{u^{-3/2}}{-3/2} = \frac{-2}{3}\frac{1}{(x-1)^{3/2}} + C$$

$$u = x-1; \; dx = du$$

$$\int_{2}^{\infty} \frac{1}{(x-1)^{5/2}}\,dx = \lim_{t\to\infty} \int_{2}^{t} \frac{1}{(x-1)^{5/2}}\,dx$$

$$= \lim_{t\to\infty}\left(\frac{-2}{3}\frac{1}{(t-1)^{3/2}} + \frac{2}{3}\frac{1}{(2-1)^{5/2}}\right)$$

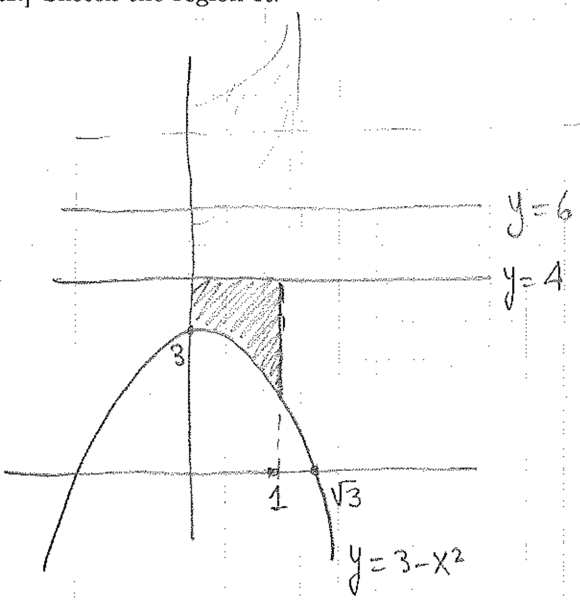
$$= \frac{2}{3}$$

*

6. Let R denote the region in the xy-plane enclosed by

$$y = 3 - x^2; \ y = 4; \ 0 \leq x \leq 1.$$

a). [2 pts.] Sketch the region R.



b). [4 pts.] Set up an integral to represent the volume of the solid obtained by rotating the region R about the line $y = 4$. *Do not compute the numerical value.*

$$V = \int_0^1 \pi (1+x^2)^2 \, dx$$

$$R = 4 - y = 4 - (3 - x^2) = 1 + x^2$$

c). [4 pts.] Set up an integral to represent the volume of the solid obtained by rotating the region R about the line $y = 6$. *Do not compute the numerical value.*

$$V = \int_0^1 \left(\pi (3+x^2)^2 - \pi \cdot 4 \right) dx$$

$$R_{out} = 6 - (3 - x^2) = 3 + x^2$$

$$R_{in} = 2$$

*

7. Determine if the series below converges or diverges. In the case of convergence, find the exact limit of the series.

a). [5 pts.] $S = \sum_{n=1}^{\infty} 2^{2n} \cdot 5^{1-n}$.

$$= \sum_{n=1}^{\infty} 4^n \cdot \frac{1}{5^{n-1}} = \sum_{n=1}^{\infty} 4 \cdot \left(\frac{4}{5}\right)^{n-1} = \frac{4}{1-\frac{4}{5}} = \boxed{20}$$

geometric w/ $a = 4$, $r = \frac{4}{5}$; $|r|<1 \Rightarrow$ convergent

b). [5 pts.] $S = \sum_{n=1}^{\infty} (\cos(8))^{-n}$. $= \sum_{n=1}^{\infty} \left(\frac{1}{\cos(8)}\right)^n$

Geometric w/ $r = \frac{1}{\cos(8)}$, $|r|>1 \Rightarrow$ divergent

★

8. Suppose that for a function f we have

$$f^{(n)}(2) = \frac{(-1)^n n!}{5^n(n+2)},$$

for all integers $n \geq 0$.

a). [2 pts.] Find the Taylor expansion of the function f centered at $a = 2$.

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{5^n(n+2)} (x-2)^n$$

b). [8 pts.] Find the radius of convergence and the interval of convergence for the power series you found in part a).

Ratio Test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-2)^{n+1}}{5^{n+1}(n+3)} \cdot \frac{5^n(n+2)}{(x-2)^n}\right| = \frac{n+2}{n+3} \cdot \frac{1}{5}|x-2| \xrightarrow[n\to\infty]{} \frac{1}{5}|x-2| < 1$$

$$|x-2| < 5 \quad \Rightarrow \boxed{R=5}$$

Endpoints: $-5 < x-2 < 5$

$-3 < x < 7$

$x=7$: $\sum_{n=0}^{\infty} \frac{(-1)^n}{5^n(n+2)} 5^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+2}$ convergent (alternating harmonic)

$x=-3$: $\sum_{n=0}^{\infty} \frac{1}{n+2}$ divergent (harmonic)

$$\boxed{I = (-3, 7]}$$

★

9. Find the arclength of the curve

$$y = \frac{x^2}{8} - \ln(x)$$

for $x \in [2,3]$.

$$y' = \frac{2x}{8} - \frac{1}{x} = \frac{x}{4} - \frac{1}{x}$$

$$L = \int_2^3 \sqrt{1 + \left(\frac{x}{4} - \frac{1}{x}\right)^2}\, dx = \int_2^3 \sqrt{1 + \frac{x^2}{16} - \frac{1}{2} + \frac{1}{x^2}}\, dx$$

$$= \int_2^3 \sqrt{\frac{x^2}{16} + \frac{1}{2} + \frac{1}{x^2}}\, dx$$

$$= \int_2^3 \sqrt{\left(\frac{x}{4} + \frac{1}{x}\right)^2}\, dx$$

$$= \int_2^3 \left(\frac{x}{4} + \frac{1}{x}\right) dx$$

$$= \left(\frac{x^2}{8} + \ln x\right)\Big|_2^3$$

$$= \frac{9}{8} + \ln 3 - \frac{4}{8} - \ln 2$$

$$= \boxed{\frac{5}{8} + \ln\frac{3}{2}}$$

10. a). [3 pts.] If a is a real number, what is

$$\sqrt{a^2} = ? \quad |a|$$

b). [7 pts.] Find the integral

$$\int x^3\sqrt{x^2+1}\,dx. \quad = \int x^2\sqrt{x^2+1}\cdot x\,dx \quad = \int (u-1)\sqrt{u}\cdot \tfrac{1}{2}du$$

$u = x^2+1 \Rightarrow x^2 = u-1$

$du = 2x\,dx \Rightarrow \frac{1}{2}du = x\,dx$

$$= \frac{1}{2}\int\left(u^{3/2} - u^{1/2}\right)du$$

$$= \frac{1}{2}\left(\frac{u^{5/2}}{5/2} - \frac{u^{3/2}}{3/2}\right) + C$$

$$= \boxed{\frac{1}{5}(x^2+1)^{5/2} - \frac{1}{3}(x^2+1)^{3/2} + C}$$

<u>OR</u>: $u = \sqrt{x^2+1} \Rightarrow u^2 = x^2+1$

$du = \frac{x}{\sqrt{x^2+1}}dx \Rightarrow x\,dx = u\,du$

$$\int x^3\sqrt{x^2+1}\,dx = \int (u^2-1)u\cdot u\,du = \int (u^4 - u^2)\,du$$

$$= \frac{u^5}{5} - \frac{u^3}{3} + C$$

$$= \frac{1}{5}(x^2+$$