1. a). [3 pts.] Rewrite the Riemann sum below as a definite integral over the interval $[0, \pi]$:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(x_i^* \cos(x_i^*)\right) \Delta x = \int_{\boxed{0}}^{\boxed{\pi}} \boxed{x\cos(x)}\, dx.$$

b). [4 pts.] Compute the definite integral you obtained in part a).

$$\int_0^{\pi} x\cos(x)\,dx = \underbrace{x\sin(x)\Big|_0^{\pi}}_{0} - \int_0^{\pi} \sin(x)\,dx$$

$u = x \quad dv = \cos(x)\,dx$

$du = dx \quad v = \sin(x)$

$$= \cos(x)\Big|_0^{\pi} = \cos(\pi) - \cos(0) = \boxed{-2}$$

c). [3 pts.] Find the derivative of the function $g(x)$ given by

$$g(x) := \int_{\sqrt{x}}^{7} \arctan(e^t)\, dt. = -\int_7^{\sqrt{x}} \arctan(e^t)\,dt$$

$$g'(x) = -\arctan\left(e^{\sqrt{x}}\right) \frac{1}{2\sqrt{x}}$$

2. Find the integrals:

a). [5 pts.] $\int \frac{e^{2x}}{e^{4x}+2e^{2x}+1}\,dx = \int \frac{e^{2x}}{(e^{2x}+1)^2}\,dx = \int \frac{1}{u^2}\frac{1}{2}\,du = \frac{-1}{2u} + C$

$$= \boxed{\frac{-1}{2(e^{2x}+1)} + C}$$

$u = e^{2x}+1$

$du = 2e^{2x}\,dx$

$\frac{1}{2}du = e^{2x}dx$

b). [5 pts.] $\int \sqrt{5-x^2}\,dx = \int 5\cos^2\theta\,d\theta = \int \frac{5}{2}(1+\cos(2\theta))\,d\theta$

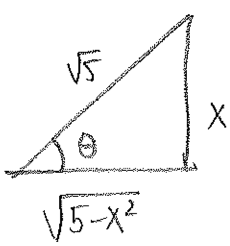
$= \frac{5}{2}\theta + \frac{5}{4}\sin(2\theta) + C$

$= \frac{5}{2}\theta + \frac{5}{2}\sin\theta\cos\theta + C$

Trig Sub: $x = \sqrt{5}\sin\theta$

$dx = \sqrt{5}\cos\theta$

$\sqrt{5-x^2} = \sqrt{5}\cos\theta$



$= \frac{5}{2}\arcsin\left(\frac{x}{\sqrt{5}}\right) + \frac{5}{2}\,\frac{x}{\sqrt{5}}\,\frac{\sqrt{5-x^2}}{\sqrt{5}} + C$

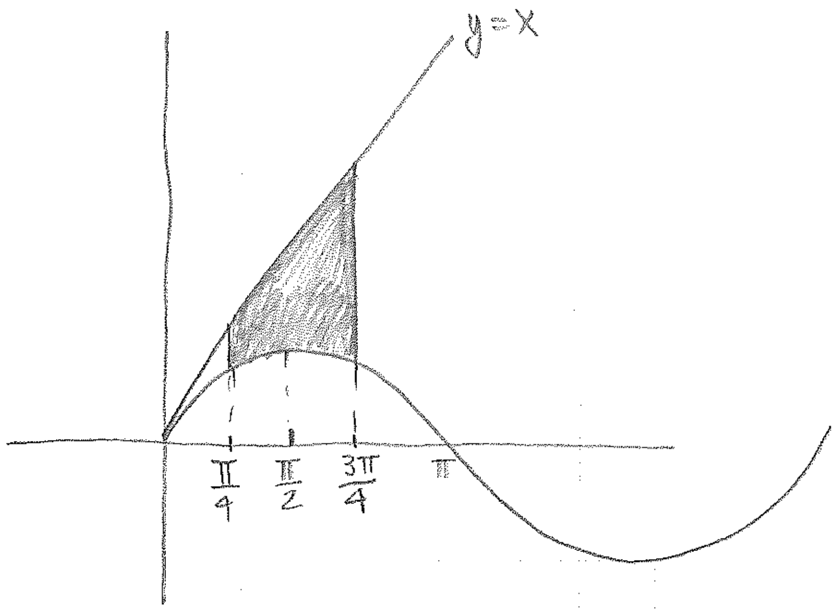
$$= \boxed{\frac{5}{2}\arcsin\left(\frac{x}{\sqrt{5}}\right) + \frac{x\sqrt{5-x^2}}{2} + C}$$

3. Consider the region R in the xy-plane bounded by the curves

$$y = \sin(x);\ \ y = x;\ \ x = \frac{\pi}{4};\ \ x = \frac{3\pi}{4}.$$

a). [2 pts.] Sketch the region R. It may be helpful to keep in mind the inequality

$$\sin(x) \leq x,\ \ \forall x \geq 0.$$



b). [4 pts.] Set up an integral for the area of the region R. *Do **not** compute the integral.*

$$A = \int_{\pi/4}^{3\pi/4} (x - \sin(x))\,dx$$

c). [4 pts.] Set up an integral for the volume of the solid obtained by rotating the region R around the x-axis. *Do **not** compute the integral.*

$$V = \int_{\pi/4}^{3\pi/4} \left(\pi x^2 - \pi \sin^2(x)\right) dx$$

$R_{out} = x$

$R_{in} = \sin(x)$

4. For each of the series below, determine if it is absolutely convergent, conditionally convergent, or divergent:

a). [5 pts.] $S = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/3}}$.

$\sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$ divergent p-series $(p = 1/3)$ $\Rightarrow$ not abs. conv.

AST: $b_n = \frac{1}{n^{1/3}}$ is decreasing and $\lim_{n\to\infty} b_n = 0$

$\Rightarrow$ conditionally convergent.

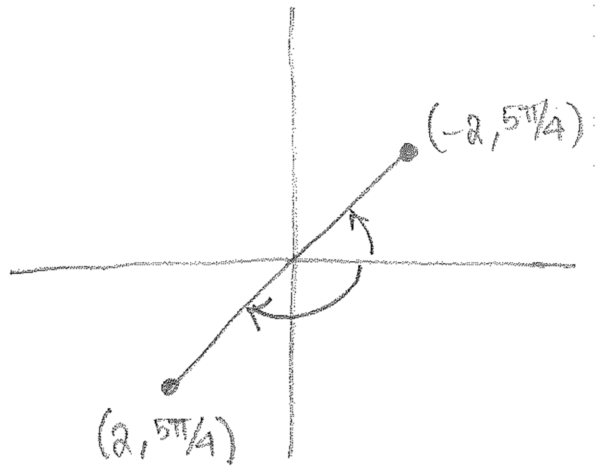
b). [5 pts.] $S = \sum_{n=1}^{\infty} \left(\frac{n+1}{n}\right)^{n^2}$.

Root Test:

$\sqrt[n]{|a_n|} = \left(\frac{n+1}{n}\right)^n \xrightarrow[n\to\infty]{} e > 1$ $\Rightarrow$ divergent by the Root Test.

5. a). [3 pts.] Plot in the Cartesian plane the point P, given by polar coordinates:

$$P: \left(-2, \frac{5\pi}{4}\right).$$



b). [2 pts.] Find two different pairs of polar coordinates (r, θ) to describe the point P, one with $r > 0$ and one with $r < 0$.

Pair 1: $(r > 0)$ $\boxed{(2, \pi/4)}$ Pair 2: $(r < 0)$ $\boxed{\left(-2, -\frac{3\pi}{4}\right)}$

c). [5 pts.] Find the Cartesian equation of the polar curve:

$$r = 5\cos\theta,$$

and plot the curve.

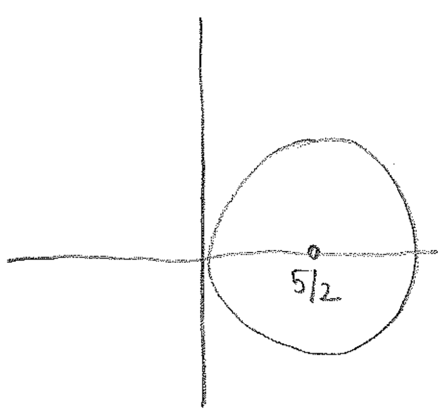
$$r^2 = 5r\cos\theta$$

$$x^2 + y^2 = 5x$$

$$x^2 - 5x + y^2 = 0$$

$$x^2 - 5x + \frac{25}{4} + y^2 = \frac{25}{4}$$

$$\left(x - \frac{5}{2}\right)^2 + y^2 = \left(\frac{5}{2}\right)^2$$



6. Find the radius of convergence and the interval of convergence for each power series below:

a). [5 pts.] $S = \sum_{n=1}^{\infty} \frac{x^n}{4n!}$.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{4(n+1)!} \frac{4n!}{x^n} \right| = \lim_{n\to\infty} \frac{|x|}{n+1} = 0 < 1, \ \forall x \in \mathbb{R}$$

$I = \mathbb{R}, \ R = \infty.$

b). [5 pts.] $S = \sum_{n=1}^{\infty} \frac{2^n (x-3)^n}{n^2}$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left| \frac{2^{n+1}(x-3)^{n+1}}{(n+1)^2} \frac{n^2}{2^n (x-3)^n} \right| = \frac{2n^2}{(n+1)^2} |x-3| \xrightarrow[n\to\infty]{} 2|x-3| < 1$$

$|x-3| < \frac{1}{2}$ $\qquad R = 1/2$

$-\frac{1}{2} < x-3 < \frac{1}{2} \Rightarrow \frac{5}{2} < x < \frac{7}{2}$

$x = \frac{5}{2}$: $\sum_{n=1}^{\infty} \frac{2^n}{n^2}\left(\frac{-1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ abs. conv.

$x = \frac{7}{2}$: $\sum_{n=1}^{\infty} \frac{2^n}{n^2}\left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ conv. p-series

$I = \left[\frac{5}{2}, \frac{7}{2}\right]$

7. Recall the geometric series:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad \forall |x| < 1.$$

a). [3 pts.] Use the geometric series above to find the Maclaurin series of the function

$$f(x) = \frac{1}{(1-x)^2}, \quad = \left(\frac{1}{1-X}\right)'$$

for $|x| < 1$.

$$= \sum_{n=1}^{\infty} nX^{n-1}, \quad |X|<1$$

b). [3 pts.] Use the result in part a). to find the Maclaurin series of the function

$$f(x) = \frac{x}{(1-x)^2}, \quad = \sum_{n=1}^{\infty} nX^n$$

for $|x| < 1$.

c). [4 pts.] Use the result in part b). to find the sum of the series

$$\sum_{n=1}^{\infty} \frac{n}{4^n} = \sum_{n=1}^{\infty} n\left(\frac{1}{4}\right)^n = \frac{1/4}{(1-1/4)^2} = \frac{1}{4} \cdot \frac{16}{9} = \frac{4}{9}$$

8. a). [6 pts.] Evaluate the integral

$$\int_2^{\infty} \frac{1}{x \ln^2(x)}\, dx.$$

$$\int \frac{1}{x \ln^2 x}\,dx = \int \frac{1}{u^2}\,du = \frac{-1}{u} + C = \frac{-1}{\ln x} + C$$

$u = \ln x$

$du = \frac{1}{x}\,dx$

$$= \lim_{t\to\infty} \int_2^t \frac{1}{x \ln^2 x}\,dx$$

$$= \lim_{t\to\infty} \left. \frac{-1}{\ln x} \right|_2^t$$

$$= \lim_{t\to\infty} \left(\frac{-1}{\ln t} + \frac{1}{\ln 2} \right)$$

$$= \frac{1}{\ln 2}$$

b). [4 pts.] What can you conclude about the series

$$S = \sum_{n=2}^{\infty} \frac{1}{n \ln^2(n)},$$

based on your results in part a).? What Test are you using for this conclusion?

The series is convergent by the Integral Test, because

$$f(x) = \frac{1}{x \ln^2(x)}$$

is decreasing, positive & continuous on $[2, \infty)$.

9. Find the arclength of the curve

$$y = \ln(\cos(x)), \ \frac{\pi}{4} \le x \le \frac{\pi}{3}.$$

$$y' = -\frac{\sin(x)}{\cos(x)} = -\tan x$$

$$L = \int_{\pi/4}^{\pi/3} \sqrt{1+\tan^2 x}\, dx = \int_{\pi/4}^{\pi/3} |\sec(x)|\, dx$$

$$= \ln|\sec x + \tan x| \Big|_{\pi/4}^{\pi/3}$$

$$\cos\frac{\pi}{3} = \frac{1}{2}$$

$$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$\boxed{= \ln\left(2+\sqrt{3}\right) - \ln\left(\sqrt{2}+1\right)}$$

10. a). [3 pts.] If a is a real number, what is

$$\sqrt{a^2} = ? \quad |a|$$

b). [7 pts.] Find the integral

$$\int \frac{x}{(x^2+1)^2} \ln(x)\, dx. \longrightarrow \frac{-\ln(x)}{2(x^2+1)} + \int \frac{1}{2x(x^2+1)}\, dx$$

$u = \ln(x)$ $\qquad dv = \dfrac{x}{(x^2+1)^2}$

$du = \dfrac{1}{x}\,dx$ $\qquad V = \dfrac{-1}{2(x^2+1)}$

Partial fractions: $\dfrac{1}{x(x^2+1)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+1} = \dfrac{1}{x} + \dfrac{-x}{x^2+1}$

$1 = Ax^2 + A + Bx^2 + Cx$

$1 = (A+B)x^2 + Cx + A$

$A = 1$

$A+B=0 \Rightarrow B = -1$

$C = 0$

$$\Rightarrow \int \frac{x}{(x^2+1)^2} \ln(x)\, dx = \frac{-\ln(x)}{2(x^2+1)} + \frac{1}{2}\int\left(\frac{1}{x} - \frac{x}{x^2+1}\right) dx$$

$$= \boxed{\frac{-\ln(x)}{2(x^2+1)} + \frac{1}{2}\ln(x) - \frac{1}{4}\ln(x^2+1) + C}$$