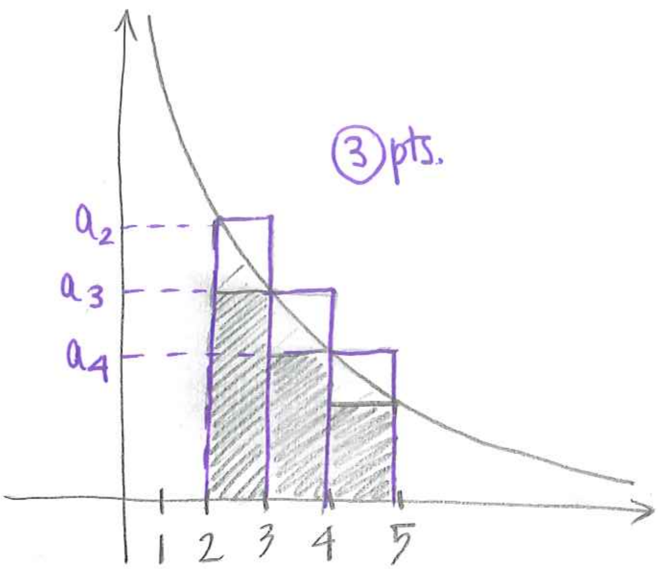
(9)

1. Suppose f is a continuous, positive, and decreasing function for $x \geq 1$. For every integer $n \geq 1$, let $a_n = f(n)$. By drawing a picture, arrange the following three quantities in increasing order:

$$\int_2^5 f(x)\,dx; \qquad \sum_{n=2}^{4} a_n; \qquad \sum_{n=3}^{5} a_n.$$



③ pts.

$\sum_{n=2}^{4} a_n = a_2 + a_3 + a_4$

(in purple, all rectangles are above the graph).

$\sum_{n=3}^{5} a_n = a_3 + a_4 + a_5$

(shaded, all rectangles lie below the graph).

$$\boxed{\sum_{n=3}^{5} a_n \leq \int_2^5 f(x)\,dx \leq \sum_{n=2}^{4} a_n}$$

③ pts. ③ pts.

(12) 3pts. each: 1.5pts. answer, 1.5pts. justification

▷

2. For each of the sequences below, write its limit in the box provided, if it exists, or write "dne" if the limit does not exist. **JUSTIFY your answer below for each sequence!**

a). $a_n = \dfrac{(-1)^n \cdot 2n}{n^2+1}$; $\lim\limits_{n\to\infty} a_n = \boxed{0}$

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{2n}{n^2+1} = 0 \Rightarrow \lim_{n\to\infty} a_n = 0.$$

b). $a_n = \sin(n+1)$; $\lim\limits_{n\to\infty} a_n = \boxed{\text{dne}}$

As $n\to\infty$, $\sin(n+1)$ varies indefinitely & periodically between -1 and 1.

c). $a_n = \dfrac{\sin(n+1)}{n^2+1}$; $\lim\limits_{n\to\infty} a_n = \boxed{0}$ by the Squeeze Thm.!

$$\frac{-1}{n^2+1} \leq \frac{\sin(n+1)}{n^2+1} \leq \frac{1}{n^2+1}$$

$$\xrightarrow{n\to\infty} 0 \xleftarrow{n\to\infty}$$

d). $a_n = \dfrac{2^{n+3}}{7^n}$; $\lim\limits_{n\to\infty} a_n = \boxed{0}$

$$a_n = 2^3 \cdot \left(\frac{2}{7}\right)^n \text{ and } \frac{2}{7} < 1,$$

(10)

▷

3. For each series below, determine whether or not it converges. Justify your answer, and make sure you state clearly any Series Tests you use.

(5) a). $S = \sum_{n=1}^{\infty} \frac{1}{4n-1}$.

Comparison Test with the harmonic series:

$$\frac{1}{4n-1} > \frac{1}{4n}$$

$\sum_{n=1}^{\infty} \frac{1}{4n}$ diverges (harmonic) $\Rightarrow$ S diverges by the Comparison Test.

OR Limit Comparison Test with the harmonic series:

$$a_n = \frac{1}{4n-1}; \quad b_n = \frac{1}{n}$$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{4n-1} = \frac{1}{4} > 0 \Rightarrow S$ diverges by LCT, since $\sum \frac{1}{n}$ diverges (harmonic)

(5) b). $S = \sum_{n=1}^{\infty} \frac{n^2}{2n^7-1}$.

LCT with p-series:

$$a_n = \frac{n^2}{2n^7-1}; \quad b_n = \frac{1}{n^5}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^7}{2n^7-1} = \frac{1}{2} > 0 \Rightarrow$$

$\Rightarrow$ Since $\sum b_n$ converges (p-series w/ $p=5$),
S also converges by LCT.

Stating Test – (1)
Using Test Correctly – (2)
Correct Conclusion – (2)

(10)

▷

4. Determine whether the series below is absolutely convergent, conditionally convergent, or divergent. Justify your answer, and make sure you state clearly any Series Tests you use.

$$S = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/3}}.$$

$\sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$ is divergent (p-series w/ $p = \frac{1}{3} < 1$)
2 1

$\Rightarrow$ S is not absolutely convergent.

$S = \sum_{n=1}^{\infty} (-1)^n b_n$, where $b_n = \frac{1}{n^{1/3}}$

* $\lim_{n \to \infty} b_n = 0$ 2

* b_n is decreasing: $\frac{1}{(n+1)^{1/3}} < \frac{1}{n^{1/3}}$ 2

$\Rightarrow$ by the Alternating Series Test, S is convergent $\Rightarrow$
1

S is conditionally convergent.
2

(8)

▷

5. Determine if each of the series below converges or diverges, and in case of convergence find its limit. Justify your answer, and make sure you state clearly any Series Tests you use.

a). $S = \sum_{n=1}^{\infty} 2^{2n} \cdot 5^{1-n}$. $= \sum_{n=1}^{\infty} 4^n \cdot \frac{5}{5^n}$

(6)

$= \sum_{n=1}^{\infty} 5 \cdot \left(\frac{4}{5}\right)^n$

$= \sum_{n=1}^{\infty} 4 \left(\frac{4}{5}\right)^{n-1}$

bring to this form (or to $\sum_{n=0}^{\infty} 4 \cdot \left(\frac{4}{5}\right)^n$) (2)

Geometric with $a = 4$, $r = \frac{4}{5}$ (2)

$\Rightarrow$ convergent since $|r| < 1$

$= \frac{4}{1 - \frac{4}{5}}$ (2)

$= \frac{4}{\frac{1}{5}} = \boxed{20}$

b). $S = \sum_{n=1}^{\infty} (\sin(3))^{-n}$. $= \sum_{n=1}^{\infty} \left(\frac{1}{\sin(3)}\right)^n$

(2)

Geometric with $r = \frac{1}{\sin(3)}$ (1)

$\Rightarrow$ divergent since $|r| > 1$.

(2)

▷

6. Determine if each of the series below converges or diverges. Justify your answer, and make sure you state clearly any Series Tests you use.

a). $S = \sum_{n=1}^{\infty} \frac{2^n}{n^{10}}$. (8)

Ratio Test : $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{2^{n+1}}{(n+1)^{10}} \cdot \frac{n^{10}}{2^n}\right| = \frac{2n^{10}}{(n+1)^{10}} \xrightarrow[n\to\infty]{} 2 > 1$

=> Series diverges by the Ratio Test.

OR: $\lim_{n\to\infty} \frac{2^n}{n^{10}} = \infty$ $\left(\frac{\text{exponential}}{\text{polynomial}}\right)$ => S diverges by the Test for Divergence.

b). $S = \sum_{n=1}^{\infty} \arctan(10n)$. (2)

$\lim_{n\to\infty} \arctan(10n) = \frac{\pi}{2} \neq 0$

=> S diverges by the Test for Divergence.

(11)

7. Find the Maclaurin series of each function below, and state the interval for x where the expansion holds true.

(7) a). $f(x) = \frac{x}{4+x^2}$.

$$= \frac{x}{4} \cdot \frac{1}{1+\frac{x^2}{4}}$$

$$= \frac{x}{4} \cdot \frac{1}{1-\left(-\frac{x^2}{4}\right)}$$ ← bring to this form: (3)

apply geometric correctly: (2)

$$= \frac{x}{4} \cdot \sum_{n=0}^{\infty} \left(-\frac{x^2}{4}\right)^n, \quad \text{for all } \left|\frac{x^2}{4}\right| < 1 \; ; \; |x|^2 < 4; \; |x| < 2$$

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{4^{n+1}}, \quad \forall\, |x| < 2}$$

final form (1) range (1)

(4) b). $f(x) = e^{4x} - e^x$.

$$= \sum_{n=0}^{\infty} \frac{(4x)^n}{n!} - \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \forall x \in \mathbb{R}$$ ← apply Maclaurin Series correctly (2)

$$= \sum_{n=0}^{\infty} \left(\frac{4^n x^n}{n!} - \frac{x^n}{n!}\right)$$

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{4^n - 1}{n!} x^n, \quad \forall x \in \mathbb{R}}$$

final form: (1) range (1)

8. Suppose that for a function $f(x)$, we have that

$$f^{(n)}(3) = \frac{(-1)^n \cdot n!}{4^n(n+1)},$$

for all integers $n \geq 0$.

(3) a). Find the Taylor series of $f(x)$ centered at 3.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(3)}{n!}(x-3)^n = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot n!}{4^n(n+1)n!}(x-3)^n$$

← apply formula (1)
← simplify n! (1)

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{4^n(n+1)}(x-3)^n}$$

← final form (1)

b). Find the radius of convergence and the interval of convergence of the series you found in part a).

(11)

Ratio Test: $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-3)^{n+1}}{4^{n+1}(n+2)} \cdot \frac{4^n(n+1)}{(x-3)^n}\right|$

← apply Ratio Test (1)
← simplify correctly (2)
limit (1)

$$= \frac{n+1}{4(n+2)}|x-3| \xrightarrow[n\to\infty]{} \frac{1}{4}|x-3| < 1$$

$\Rightarrow$ Convergent for $|x-3|<4$ $\Rightarrow$ $\boxed{R=4}$ (1)

$-4 < x-3 < 4$

$-1 < x < 7$ (1)

Endpoints: $x=-1$: (2)

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{4^n(n+1)}(-4)^n = \sum_{n=0}^{\infty} \frac{\cancel{4^n}}{\cancel{4^n}(n+1)} = \sum_{n=0}^{\infty} \frac{1}{n+1}$$

divergent (harmonic)

$x=7$: (2)

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{\cancel{4^n}(n+1)}\cancel{4^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$$

convergent (alternating harmonic).

$\Rightarrow$ $\boxed{I = (-1, 7]}$ (1)

(1,5) each

(12)

9. TRUE or FALSE: For each following statement, circle below it T for "true" or F for "false."

a). If the sequence a_n satisfies $\lim_{n\to 0} a_n = 0$, then the series $\sum_{n=1}^{\infty} a_n$ converges.

typo

T F

Harmonic: $\sum \frac{1}{n}$ diverges, and $\frac{1}{n} \to 0$.

b). If the series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to 0} a_n = 0$.

typo

T F

Test for Divergence.

c). If the series $\sum_{n=1}^{\infty} a_n$ diverges, then the sequence a_n cannot converge to 0.

T F

Harmonic: $\sum \frac{1}{n}$ diverges, and $\frac{1}{n} \to 0$.

d). If the series $\sum_{n=1}^{\infty} a_n$ converges, then its sequence of partial sums must also converge.

T F

$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n$ by definition

e). If the series $\sum_{n=1}^{\infty} |a_n|$ converges, then the series $\sum_{n=1}^{\infty} a_n$ must also converge.

T F

Abs. conv. $\Rightarrow$ conv.

f). The Comparison Test may be applied directly to the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$.

T F

Comparison Test only applies to series w/ positive terms.

g). The Ratio Test is inconclusive for the harmonic series.

T F

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{n}{n+1} = 1$

h). The interval of convergence of the power series $\sum_{n=0}^{\infty} c_n x^n$ could be $(-1, 8)$.

T F

Power series centered at 0, interval of convergence must be of the form $(-R, R)$.

(10)

10. Prove, using the $\epsilon - N(\epsilon)$ limit definition, that

$$\lim_{n\to\infty} (\sqrt{n+2} - \sqrt{n}) = 0.$$

Complete the guided proof below, and make sure to write any calculations you make under the proof or on the side.

Proof. The sequence we are dealing with is

(3) $$a_n := \boxed{\sqrt{n+2} - \sqrt{n}}.$$

We must show that:

For all $\epsilon > 0$, there is a number $N = N(\epsilon)$ such that

(3) $$\boxed{\left|\sqrt{n+2} - \sqrt{n}\right| < \varepsilon} \quad (\star),$$

for all $n > N(\epsilon)$.

The expression in $(\star)$ can be further analyzed to be the same as:

(2) $$\boxed{\frac{2}{\sqrt{n+2}+\sqrt{n}} < \varepsilon}$$

Further simplifications show that this is true for all n such that:

(1) $$\boxed{\frac{1}{\varepsilon^2}} < n.$$

(1/2) (1/2)

Therefore, for all $n > \boxed{1/\varepsilon^2}$, we have $\boxed{|a_n| < \varepsilon}$ and the theorem is proved. □

$$\sqrt{n+2} - \sqrt{n} = \frac{n+2-n}{\sqrt{n+2}+\sqrt{n}} = \frac{2}{\sqrt{n+2}+\sqrt{n}}$$

Need: $\dfrac{2}{\sqrt{n+2}+\sqrt{n}} < \varepsilon$; $\dfrac{2}{\varepsilon} < \sqrt{n+2} + \sqrt{n}$

Put: $\dfrac{2}{\varepsilon} < \sqrt{n} + \sqrt{n} = 2\sqrt{n} < \sqrt{n+2} + \sqrt{n}$

$\dfrac{2}{\varepsilon} < 2\sqrt{n} \Rightarrow \dfrac{1}{\varepsilon^2} < n$

~~$\frac{2}{\sqrt{n}}$~~