(12) 3pts. each: 1.5 pts. answer, 1.5 pts. justification

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1. For each of the sequences below, write its limit in the box provided, if it exists, or write "dne" if the limit does not exist. **JUSTIFY your answer below for each sequence!**

a). $a_n = \cos(n+1)$; $\lim_{n\to\infty} a_n = \boxed{\text{dne}}$

As $n \to \infty$, $\cos(n+1)$ varies indefinitely & periodically between -1 & 1,

b). $a_n = \dfrac{(-1)^n \cdot n^2}{3n^3+1}$; $\lim_{n\to\infty} a_n = \boxed{0}$

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{n^2}{3n^3+1} = 0 \Rightarrow \lim_{n\to\infty} a_n = 0.$$

c). $a_n = \dfrac{\cos(n+1)}{3n^3+1}$; $\lim_{n\to\infty} a_n = \boxed{0}$ by the Squeeze Theorem.

$$\frac{-1}{3n^3+1} \leq \frac{\cos(n+1)}{3n^3+1} \leq \frac{1}{3n^3+1}$$

$$\xrightarrow{n\to\infty} 0 \xleftarrow{n\to\infty}$$

d). $a_n = \dfrac{3^{n+2}}{8^n}$; $\lim_{n\to\infty} a_n = \boxed{0}$

$a_n = 3^2 \cdot \left(\frac{3}{8}\right)^n$ and $\frac{3}{8} < 1$

(a)

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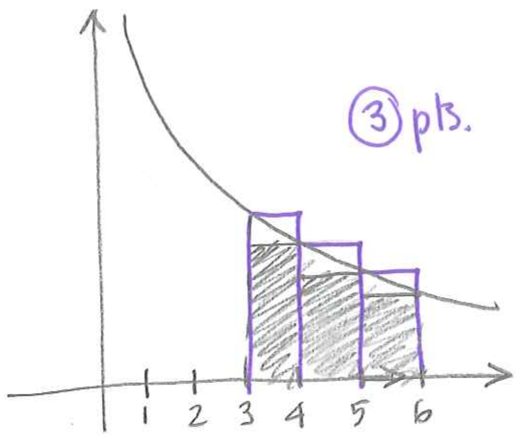
2. Suppose f is a continuous, positive, and decreasing function for $x \geq 1$. For every integer $n \geq 1$, let $a_n = f(n)$. By drawing a picture, arrange the following three quantities in increasing order:

$$\int_3^6 f(x)\,dx; \qquad \sum_{n=3}^{5} a_n; \qquad \sum_{n=4}^{6} a_n.$$



3 pts.

$$\sum_{n=3}^{5} a_n = a_3 + a_4 + a_5$$

(in purple, all rectangles above the graph).

$$\sum_{n=4}^{6} a_n = a_4 + a_5 + a_6$$

(shaded, all rectangles below the graph).

$$\sum_{n=4}^{6} a_n \leq \int_3^6 f(x)\,dx \leq \sum_{n=3}^{5} a_n$$

3 pts. 3 pts.

(10)

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3. Determine whether the series below is absolutely convergent, conditionally convergent, or divergent. Justify your answer, and make sure you state clearly any Series Tests you use.

$$S = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{2/5}}.$$

$\sum_{n=1}^{\infty} \frac{1}{n^{2/5}}$ is divergent (p-series w/ $p = \frac{2}{5} < 1$)

2 1

$\Rightarrow$ S is not abs. conv.

$S = \sum_{n=1}^{\infty} (-1)^n b_n$, where $b_n = \frac{1}{n^{2/5}}$

* $\lim_{n\to\infty} b_n = 0$ 2

* b_n is decreasing: $\frac{1}{(n+1)^{2/5}} < \frac{1}{n^{2/5}}$ 2

$\Rightarrow$ by the Alternating Series Test, S is convergent

1

$\Rightarrow$ S is conditionally convergent.

2

(10)

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4. For each series below, determine whether or not it converges. Justify your answer, and make sure you state clearly any Series Tests you use.

(5) a). $S = \sum_{n=1}^{\infty} \frac{n^3}{n^7 - 1}$.

LCT with p-series:

$$a_n = \frac{n^3}{n^7 - 1} \; ; \; b_n = \frac{1}{n^4}$$

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{n^7}{n^7 - 1} = 1 > 0$$

$\Rightarrow$ S converges by LCT, since $\sum \frac{1}{n^4}$ converges (p-series w/ $p = 4 > 1$).

b). $S = \sum_{n=1}^{\infty} \frac{1}{2n - 1}$.

(5)

$$\frac{1}{2n-1} > \frac{1}{2n}$$

$$\sum \frac{1}{2n} = \frac{1}{2} \sum \frac{1}{n} = \infty \text{ (harmonic)}$$

$\Rightarrow$ S diverges by the Comparison Test.

OR: LCT with harmonic:

$$a_n = \frac{1}{2n-1} \; ; \; b_n = \frac{1}{n}$$

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{n}{2n-1} = \frac{1}{2} > 0$$

$\Rightarrow$ Since $\sum \frac{1}{n}$ diverges (harmonic), S also diverges by LCT.

Stating Test - (1)
Using Test Correctly - (2)
Correct Conclusion - (2)

(10) *

5. Determine if each of the series below converges or diverges. Justify your answer, and make sure you state clearly any Series Tests you use.

a). $S = \sum_{n=1}^{\infty} \frac{n^{10}}{2^n}$. (8)

Ratio Test: $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^{10}}{2^{n+1}} \cdot \frac{2^n}{n^{10}}\right| = \frac{(n+1)^{10}}{n^{10}} \cdot \frac{1}{2} \xrightarrow[n\to\infty]{} \frac{1}{2} < 1$

$\Rightarrow$ S converges by the Ratio Test.

b). $S = \sum_{n=1}^{\infty} \arctan(2n)$. (2)

$\lim_{n\to\infty} \arctan(2n) = \frac{\pi}{2} \neq 0$

$\Rightarrow$ S diverges by the Test for Divergence.

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6. Determine if each of the series below converges or diverges, and in case of convergence find its limit. Justify your answer, and make sure you state clearly any Series Tests you use.

a). $S = \sum_{n=1}^{\infty} 3^{2n} \cdot 10^{1-n}.$ $= \sum_{n=1}^{\infty} 9^n \cdot \frac{1}{10^{n-1}}$

(6)

bring to this form (2)

$(\text{or } \sum_{n=0}^{\infty} 9 \cdot (\frac{9}{10})^n)$

$= \sum_{n=1}^{\infty} 9 \cdot \left(\frac{9}{10}\right)^{n-1}$

Geometric w/ $a=9$, $r=\frac{9}{10}$ (2)

=> convergent since $|r|<1$.

$= \frac{9}{1-\frac{9}{10}}$

$= \frac{9}{\frac{1}{10}}$ (2)

$= \boxed{90}$

b). $S = \sum_{n=1}^{\infty} \left(\cos(3)\right)^{-n}.$ $= \sum_{n=1}^{\infty} \left(\frac{1}{\cos(3)}\right)^n$

(2)

Geometric with $r = \frac{1}{\cos(3)}$ (1)

=> divergent since $|r|>1$.

(2)

(14)

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7. Suppose that for a function $f(x)$, we have that

$$f^{(n)}(2) = \frac{(-1)^n \cdot n!}{3^n(n+1)},$$

for all integers $n \geq 0$.

a): Find the Taylor series of $f(x)$ centered at 2.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot n!}{3^n(n+1)n!}(x-2)^n$$ ← apply formula 1 ← simplify n! 1

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{3^n(n+1)}(x-2)^n}$$ ← final form 1

b). Find the radius of convergence and the interval of convergence of the series you found in part a).

Ratio Test: $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-2)^{n+1}}{3^{n+1}(n+2)} \cdot \frac{3^n(n+1)}{(x-2)^n}\right|$ ← apply Ratio Test 1 ← simplify correctly 2 ← limit 1

$$= \frac{n+1}{n+2}\,\frac{1}{3}\,|x-2| \xrightarrow[n\to\infty]{} \frac{|x-2|}{3} < 1$$

$\Rightarrow$ Convergent for $|x-2| < 3 \Rightarrow \boxed{R=3}$ 1

$-3 < x-2 < 3$

$-1 < x < 5$ 1

Endpoints: $x=-1$: 2 $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n(-3)^n}{3^n(n+1)} = \sum_{n=0}^{\infty} \frac{1}{n+1}$ divergent (harmonic)

$x=5$: 2 $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \cancel{3^n}}{\cancel{3^n}(n+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$ convergent (alt. harmonic)

$\Rightarrow \boxed{I = (-1, 5]}$ 1

(11)

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8. Find the Maclaurin series of each function below, and state the interval for x where the expansion holds true.

(7) a). $f(x) = \frac{x}{3+x^2}$.

$$= \frac{x}{3} \cdot \frac{1}{1+\frac{x^2}{3}}$$

$$= \frac{x}{3} \cdot \frac{1}{1-\left(-\frac{x^2}{3}\right)}$$ ← bring to this form (3)

apply geometric correctly (2)

$$= \frac{x}{3} \sum_{n=0}^{\infty} \left(-\frac{x^2}{3}\right)^n, \text{ for all } \left|\frac{x^2}{3}\right| < 1;\ |x|^2 < 3;\ |x| < \sqrt{3}$$

$$\Rightarrow \boxed{f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{3^{n+1}}, \quad \forall\ |x| < \sqrt{3}}$$

final form (1) range (1)

(4) b). $f(x) = x^2 \ln(1+x^3)$.

$$f(x) = x^2 \cdot \sum_{n=1}^{\infty} \frac{(-1)^{n-1} (x^3)^n}{n}, \quad \forall\ |x^3| < 1;\ |x| < 1.$$ ← Apply Maclaurin series correctly (2)

$$\Rightarrow \boxed{f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{3n+2}}{n}, \quad \forall\ |x| < 1}$$

final form (1) range (1)

(12) 1.5 each

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9. TRUE or FALSE: For each following statement, circle below it T for "true" or F for "false."

a). The Comparison Test may be applied directly to the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$.

T F

Comp. Test only applies to Series w/ positive terms.

b). The interval of convergence of the power series $\sum_{n=0}^{\infty} c_n x^n$ could be $(-1, 8)$.

T F

A power series centered at 0 must have interval of conv. in the form [(−R, R)]

c). The Ratio Test is inconclusive for the harmonic series.

T F

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{n}{n+1} = 1$

d). If the series $\sum_{n=1}^{\infty} |a_n|$ converges, then the series $\sum_{n=1}^{\infty} a_n$ must also converge.

T F

Abs. conv. $\Rightarrow$ conv.

e). If the series $\sum_{n=1}^{\infty} a_n$ converges, then its sequence of partial sums must also converge.

T F

$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n$

f). If the sequence a_n satisfies $\lim_{n\to 0} a_n = 0$, then the series $\sum_{n=1}^{\infty} a_n$ converges.

typo

T F

Harmonic: $\lim_{n\to\infty} \frac{1}{n} = 0$, $\sum \frac{1}{n} = \infty$

g). If the series $\sum_{n=1}^{\infty} a_n$ diverges, then the sequence a_n cannot converge to 0.

T F

Harmonic: $\sum \frac{1}{n}$ diverges, $\frac{1}{n} \to 0$.

h). If the series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to 0} a_n = 0$.

typo

T F

Test for Divergence.

(10)

10. Prove, using the $\epsilon - N(\epsilon)$ limit definition, that

$$\lim_{n\to\infty} (\sqrt{n+3} - \sqrt{n}) = 0.$$

Complete the guided proof below, and make sure to write any calculations you make under the proof or on the side.

Proof. The sequence we are dealing with is

(3) $a_n := \boxed{\sqrt{n+3} - \sqrt{n}}$.

We must show that:

For all $\epsilon > 0$, there is a number $N = N(\epsilon)$ such that

(3) $\boxed{\left|\sqrt{n+3} - \sqrt{n}\right| < \varepsilon}$ $(\star)$,

for all $n > N(\epsilon)$.

The expression in $(\star)$ can be further analyzed to be the same as:

(2) $\boxed{\dfrac{3}{\sqrt{n+3}+\sqrt{n}} < \varepsilon}$

Further simplifications show that this is true for all n such that:

(1) $\boxed{\dfrac{9}{4\varepsilon^2}} < n.$

(1/2) (1/2)

Therefore, for all $n > \boxed{9/4\varepsilon^2}$, we have $\boxed{|a_n| < \varepsilon}$ and the theorem is proved. □

$$\left|\sqrt{n+3} - \sqrt{n}\right| = \frac{n+3-n}{\sqrt{n+3}+\sqrt{n}} = \frac{3}{\sqrt{n+3}+\sqrt{n}}$$

Need: $\dfrac{3}{\sqrt{n+3}+\sqrt{n}} < \varepsilon$; $\dfrac{3}{\varepsilon} < \sqrt{n+3}+\sqrt{n}$

Put: $\underbrace{\dfrac{3}{\varepsilon} < \sqrt{n}+\sqrt{n}}= 2\sqrt{n} < \sqrt{n+3}+\sqrt{n}$

$\dfrac{3}{\varepsilon} < 2\sqrt{n}$; $\dfrac{3}{2\varepsilon} < \sqrt{n}$; $\dfrac{9}{4\varepsilon^2} < n$