Name: ____

April 13th, 2016.
Math 2551; Sections L1, L2, L3.
Georgia Institute of Technology
Exam 3

I commit to uphold the ideals of honor and integrity by refusing to betray the trust bestowed upon me as a member of the Georgia Tech community. By signing my name below I pledge that I have neither given nor received help on this exam.

Pledged: ____

Problem	Possible Score	Earned Score
1	16	
2	16	
3	18	
4	18	
5	14	
6	18	
Total	100	

Remember that you must SHOW YOUR WORK to receive credit!

Good luck!

Angle θ $(0 \le \theta \le \pi)$ between vectors $\mathbf{u}$ and $\mathbf{v}$:

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}.$$

Vector Projection of $\mathbf{u}$ onto $\mathbf{v} \neq 0$:

$$\text{Proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v} = |\mathbf{u}\cos\theta|\frac{\mathbf{v}}{|\mathbf{v}|}.$$

Distance from a point S to a line L going through P and parallel to $\mathbf{v}$:

$$d = \frac{|\overrightarrow{PS}\times\mathbf{v}|}{|\mathbf{v}|}$$

Length of a smooth curve C: $\mathbf{r}(t)$, traced exactly once as $a \le t \le b$:

$$L = \int_a^b |\mathbf{v}(t)|\,dt.$$

TNB Frame:

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|};\quad \mathbf{N} = \frac{{}^{d\mathbf{T}}/_{ds}}{\kappa} = \frac{{}^{d\mathbf{T}}/_{dt}}{|{}^{d\mathbf{T}}/_{dt}|};\quad \mathbf{B} = \mathbf{T}\times\mathbf{N}.$$

Curvature:

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right|.$$

Tangential and Normal Components of Acceleration:

$$\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N};$$

$$a_T = \frac{d^2s}{dt^2} = \frac{d}{dt}|\mathbf{v}(t)|;$$

$$a_N = \kappa\left(\frac{ds}{dt}\right)^2 = \kappa|\mathbf{v}(t)|^2 = \sqrt{|\mathbf{a}|^2 - a_T^2}.$$

Torsion:

$$\tau = -\frac{d\mathbf{B}}{ds}\cdot\mathbf{N}.$$

Directional Derivative of f at P_0 in the direction of the unit vector $\mathbf{u}$:

$$(D_{\mathbf{u}}f)_{P_0} = (\nabla f)_{P_0}\cdot\mathbf{u}.$$

Spherical Coordinates: (ρ,ϕ,θ):

$$0 \le \phi \le \pi;\ \ 0 \le \theta \le 2\pi;$$

$$x = \rho\sin\phi\cos\theta;\ \ y = \rho\sin\phi\sin\theta;\ \ z = \rho\cos\phi;$$

$$\text{Jacobian: } dV \mapsto \rho^2\sin\phi\,d\rho\,d\phi\,d\theta.$$

Green's Theorem in the Plane:

$$\oint_C \vec{F}\cdot\vec{n}\,ds = \oint_C M\,dy - N\,dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)dA;$$

$$\oint_C \vec{F}\cdot\vec{T}\,ds = \oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dA$$

Area with Green's Theorem:

$$\text{Area}(R) = \frac{1}{2}\oint_C x\,dy - y\,dx.$$

Surface Differential on Parametric Surface S : $\mathbf{r}(u,v)$; $(u,v)\in R$:

$$d\sigma = |\mathbf{r}_u \times \mathbf{r}_v|\ d(u,v)$$

Unit Normal Field on Parametric Surface S : $\mathbf{r}(u,v)$; $(u,v)\in R$:

$$\mathbf{n} = \pm\frac{\mathbf{r}_u\times\mathbf{r}_v}{|\mathbf{r}_u\times\mathbf{r}_v|}$$

Surface Differential on Implicitly Defined (Level Surface) $f(x,y,z) = c$, over shadow region R in a coordinate plane:

$$d\sigma = \frac{|\nabla f|}{|\nabla f\cdot\mathbf{p}|}\,dA,$$

where $\mathbf{p}$ is one of $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$.
Unit Normal Field on Implicitly Defined (Level Surface) $f(x,y,z) = c$, over shadow region R in a coordinate plane:

$$\mathbf{n} = \pm\frac{\nabla f}{|\nabla f|}.$$

Parametrized Sphere of radius R, centered at the origin: $0 \le \phi \le \pi$; $0 \le \theta \le 2\pi$;

$$\mathbf{r}(\phi,\theta) = R\left\langle\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi\right\rangle;$$

$$\mathbf{r}_\phi\times\mathbf{r}_\theta = R^2\left\langle\sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \sin\phi\cos\phi\right\rangle;$$

$$|\mathbf{r}_\phi\times\mathbf{r}_\theta| = R^2\sin\phi.$$

Stokes' Theorem:

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,d\sigma,$$

(with the appropriate assumptions on C, S and $\mathbf{F}$.)

Divergence Theorem:

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iiint_D \nabla\cdot\mathbf{F}\,dV,$$

(with the appropriate assumptions on S, D and $\mathbf{F}$.)

1. The following field:

$$\vec{F}(x,y,z) = \left\langle y^3 \cos(xy^3),\quad 3xy^2 \cos(xy^3) + e^{z^2},\quad 2yze^{z^2} \right\rangle$$

is conservative.

a). Find a potential function f for this field.

b). Compute $\int_C \vec{F} \cdot d\vec{r}$, where C is a smooth path from $(\pi, 1, 0)$ to $(0, 2, 1)$.

2. Compute the integral:

$$\int_0^1 \int_0^2 \int_{4y}^8 \frac{\cos(x^2)}{\sqrt{z}}\, dx\, dy\, dz.$$

3. (a). Compute

$$\oint_C (2y + \sqrt{1 + x^5})\, dx + (5x - e^{y^2})\, dy$$

where C is the positively oriented circle $x^2 + y^2 = 4$.

(b). Compute the outward flux of the field

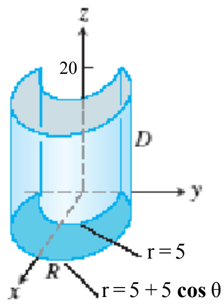
$$\mathbf{F}(x, y) = \langle x^2, y^2 \rangle$$

across the positively oriented closed curve C, consisting of the upper half of the circle $x^2 + y^2 = 4$, followed by the line segment joining $(-2, 0)$ and $(2, 0)$ along the x-axis.

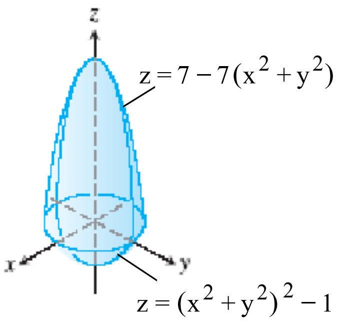
4. (a). Use cylindrical coordinates to set up the triple integral which gives the volume of the solid vertical cylinder whose:

- base is the region in the xy-plane that lies inside the cardioid $r = 5 + 5\cos\theta$ and outside the circle $r = 5$,
- top lies in the plane $z = 20$.

You do not need to compute the volume.



(b). Use cylindrical coordinates to set up the triple integral which gives the volume of the solid bounded by the surfaces $z = 7 - 7(x^2 + y^2)$ and $z = (x^2 + y^2)^2 - 1$. You do not need to compute the volume.

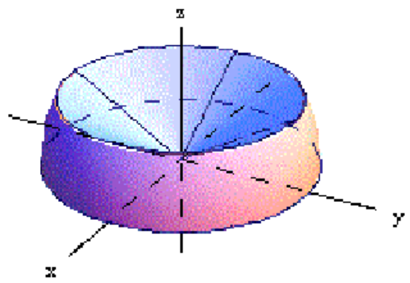


5. Compute the line integral

$$\int_C (x^2 - y)\, ds,$$

where C is the portion of the circle $x^2 + y^2 = 4$ from $(\sqrt{2}, \sqrt{2})$ to $(0, 2)$.

6. (a). Use spherical coordinates to set up the integral which gives the volume of the solid bounded below by the xy-plane, bounded laterally by the sphere $\rho = 8$, and bounded above by the cone $\phi = \frac{\pi}{6}$. You do not need to compute the volume.



(b). Use spherical coordinates to set up the integral which gives the volume of the smaller cap cut by $z = 5$ from the sphere $x^2 + y^2 + z^2 = 100$. You do not need to compute the volume.

Bonus - 5 points: Set up the volume integral in part (b) above in *cylindrical coordinates.*