Name: Solutions

May 6$^{\text{th}}$, 2016.
Math 2551; Sections L1, L2, L3.
Georgia Institute of Technology
Final Exam

I commit to uphold the ideals of honor and integrity by refusing to betray the trust bestowed upon me as a member of the Georgia Tech community. By signing my name below I pledge that I have neither given nor received help on this exam.

Pledged: ______________________________

Problem	Possible Score	Earned Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
11	10	
12	10	
13	10	
14	10	
Total	140	

Remember that you must SHOW YOUR WORK to receive credit!

Good luck!

1. Find the minimum and maximum value of $f(x,y) = x^2 + y^2$ subject to the constraint

$$x^2 - 8x + y^2 - 4y = 0.$$

Indicate the points where these extreme values occur.

Lagrange Multipliers: $f(x,y) = x^2 + y^2$ $\quad$ $\begin{cases} \nabla f = \lambda(\nabla g) \\ g = 0 \end{cases}$ (2pts.) Setup

$g(x,y) = x^2 - 8x + y^2 - 4y$

$\nabla f = \langle 2x, 2y \rangle$ $\qquad$ (2pts.)

$\nabla g = \langle 2x-8, 2y-4 \rangle$

$$\begin{cases} x = \lambda(x-4) \longrightarrow (\lambda-1)x = 4\lambda \rightarrow x = \dfrac{4\lambda}{\lambda-1} \\ y = \lambda(y-2) \longrightarrow (\lambda-1)y = 2\lambda \rightarrow y = \dfrac{2\lambda}{\lambda-1} \\ x^2 - 8x + y^2 - 4y = 0 \end{cases} \quad x = 2y$$

$4y^2 - 16y + y^2 - 4y = 0$

$5y^2 - 20y = 0$

$5y(y-4) = 0$

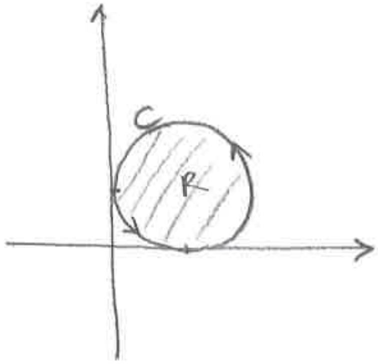
=> Extrema occur at $(0,0)$ and $(8,4)$

$f(0,0) = 0 \quad \Rightarrow \quad \boxed{\text{minimum} = 0, \text{ at } (0,0)}$ (3pts.)

$f(8,4) = 64 + 16 = 80 \quad \Rightarrow \quad \boxed{\text{maximum} = 80, \text{ at } (8,4)}$ (3pts.)

(5pts.)

2. (a). Find the work done by the field $\mathbf{F}(x,y) = \langle 2x - 5y, 5x - 2y \rangle$ in moving a particle once counter-clockwise around the curve $(x-1)^2 + (y-1)^2 = 1$.



Green : $\oint_C \vec{F} \cdot \vec{T}\, ds = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$

$M = 2x - 5y$

$N = 5x - 2y$

$= \iint_R (5 - (-5))\, dA$

$= (10) \cdot \text{Area}(R)$

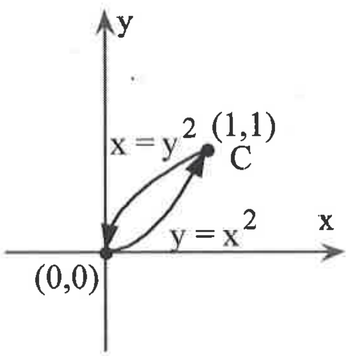
$= 10\pi$

(5pts.)

(b). Find the outward flux of the field

$$\mathbf{F}(x,y) = \langle \overset{M}{4xy + y^2},\ \overset{N}{4x - y} \rangle ,$$

across the curve C pictured below.



Green: $\oint_C \vec{F} \cdot \vec{n}\, ds = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dA$

$= \iint_R (4y - 1)\, dA$

$= \int_0^1 \int_{x^2}^{\sqrt{x}} (4y - 1)\, dy\, dx$

$= \int_0^1 (2y^2 - y) \Big|_{x^2}^{\sqrt{x}} dx$

$= \int_0^1 (2x - \sqrt{x} - 2x^4 + x^2)\, dx$

$= \left(x^2 - \frac{2}{3} x^{3/2} - \frac{2x^5}{5} + \frac{x^3}{3} \right) \Big|_0^1$

$= 1 - \frac{2}{3} - \frac{2}{5} + \frac{1}{3}$

$= \frac{2}{3} - \frac{2}{5}$

$= \frac{4}{15}$

3. Find all the critical points of the function:

$$f(x,y) = x^3 - 3xy + y^3,$$

and classify each one as a local minimum, a local maximum, or a saddle point.

(1pt.) $f_x = 3x^2 - 3y$

$f_y = -3x + 3y^2$

(1pt.) $\begin{cases} x^2 - y = 0 \\ -x + y^2 = 0 \end{cases}$ $\Rightarrow y = x^2$

$-x + x^4 = 0 \Rightarrow x(x^3 - 1) = 0 \Rightarrow \begin{array}{l} x = 0 \\ y = 0 \end{array}$ or $\begin{array}{l} x = 1 \\ y = 1 \end{array}$

Critical Points : $\boxed{(0,0)}$ & $\boxed{(1,1)}$ (4pts.)

$f_{xx} = 6x$

$f_{yy} = 6y$

$f_{xy} = -3$

$\Delta_f = f_{xx} f_{yy} - f_{xy}^2 = 36xy - 9$

$\Delta_f(0,0) < 0 \Rightarrow$ Saddle point $(0,0)$ (4pts.)

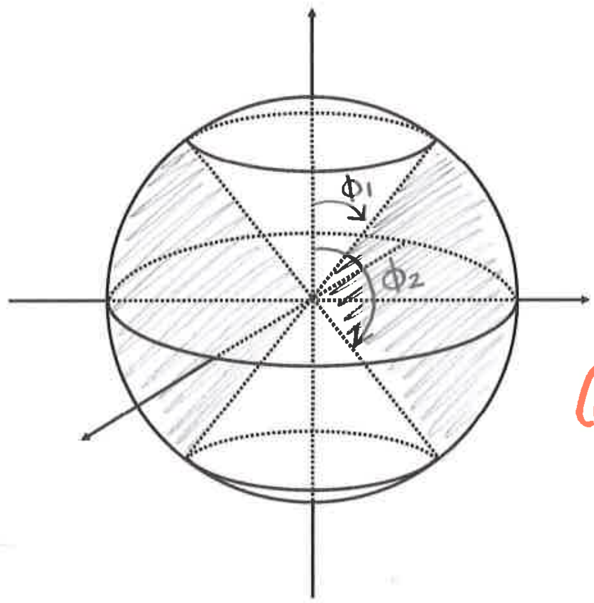
$\Delta_f(1,1) > 0$, $f_{xx}(1,1) > 0 \Rightarrow$ Local min $(1,1)$.

$\boxed{4.}$ (a). Let x be a real number. What is $\sqrt{x^2}$?

$$\sqrt{x^2} = |x|$$

(2pts.)

(b). Let D be the solid that lies *inside* the sphere $x^2 + y^2 + z^2 = 2$ and *outside* the cone $z^2 = x^2 + y^2$. Use *spherical* coordinates to set up the triple integral that gives the volume of D. You do not need to compute the volume.



Find ϕ_1, ϕ_2 (angles of cone)

Cone: $z^2 = r^2$

$\left.\begin{array}{l} z = \rho\cos\phi \\ r = \rho\sin\phi \end{array}\right\}$ in Spherical $\Rightarrow$

$\Rightarrow \cancel{\rho^2}\cos^2\phi = \cancel{\rho^2}\sin^2\phi$

(6pts.) $\Rightarrow \tan^2\phi = 1$

$\Rightarrow \tan\phi = \pm 1$

$\Rightarrow \phi_1 = \pi/4 \qquad \phi_2 = \dfrac{3\pi}{4}$

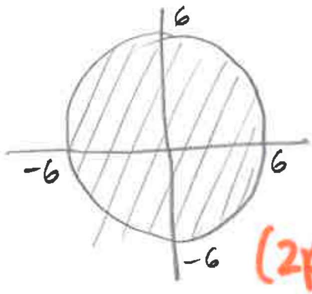
$$V = \int_0^{2\pi} \int_{\pi/4}^{3\pi/4} \int_0^{\sqrt{2}} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

(2pts.)

$\boxed{5.}$ (a). Sketch the region of integration and convert the Cartesian integral:

$$\int_{-6}^{6}\int_{-\sqrt{36-x^2}}^{\sqrt{36-x^2}} \ln(x^2+y^2+1)\,dy\,dx$$

into an equivalent polar integral. You do not need to compute the integrals.



(2pts.)

$x^2+y^2=36$

$$\int_0^{2\pi}\int_0^{6} \ln(r^2+1)\, r\,dr\,d\theta$$

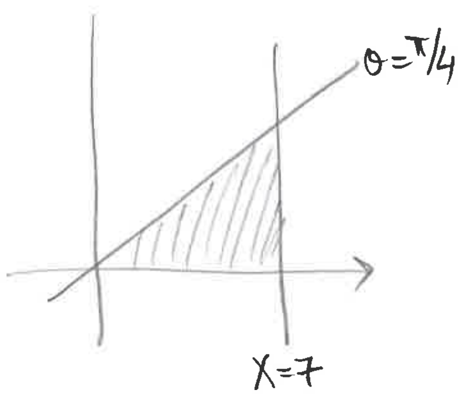
(2pts.)

(b). Sketch the region of integration and convert the polar integral:

$$\int_0^{\pi/4}\int_0^{7\sec\theta} r^5\cos^2\theta\,dr\,d\theta$$

into an equivalent Cartesian integral. You do not need to compute the integrals.

$r = 7\sec\theta \Rightarrow x = 7$



(2pts.)

$$\int_0^{7}\int_0^{x} x^2(x^2+y^2)\,dy\,dx$$

or

$$\underbrace{\int_0^{7}\int_y^{7}}_{\text{(2pts.)}} \underbrace{x^2(x^2+y^2)\,dx\,dy}_{\text{(2pts.)}}$$

6. Consider the surface S given by the parametrization:

$$\mathbf{r}(\theta, z) = \langle 2\sin(2\theta),\ 4\sin^2\theta,\ z\rangle.$$

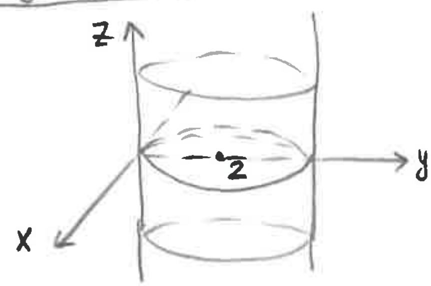
(a). Find a Cartesian equation for the surface S and sketch it. (Recall that $\sin(2\theta) = 2\sin\theta\cos\theta$).

$$\left.\begin{array}{l} x = 2\sin(2\theta) = 4\sin\theta\cos\theta \\ y = 4\sin^2\theta \\ z = z \end{array}\right\} \Rightarrow x^2+y^2 = 16\sin^2\theta = 4y$$

$$\Rightarrow x^2 + y^2 - 4y = 0$$

$$\Rightarrow \boxed{x^2+(y-2)^2 = 4} \quad \text{(3pts.)}$$

$\Rightarrow$ Cylinder w/ base $x^2+(y-2)^2=4$ (circle of radius 2 centered at $(0,2)$).



(1pt.)

(b). Find the equation of the plane tangent to the surface S at the point $P_0(\sqrt{3}, 1, 1)$ corresponding to $(\theta, z) = (\pi/6, 1)$.

Solution 1 : Using parametrization:

$$\vec{r}_\theta = \langle 4\cos(2\theta),\ 8\sin\theta\cos\theta,\ 0\rangle$$

(2pts.) $$= \langle 4\cos(2\theta),\ 4\sin(2\theta),\ 0\rangle$$

$$\vec{r}_z = \langle 0,\ 0,\ 1\rangle$$

(1pt.) $$\vec{r}_\theta \times \vec{r}_z = \langle 4\sin(2\theta), -4\cos(2\theta), 0\rangle$$

$$\vec{r}_\theta \times \vec{r}_z\Big|_{(\pi/6,1)} = \langle 4\sin(\pi/3), -4\cos(\pi/3), 0\rangle$$

(2pts.) $$= \langle 2\sqrt{3}, -2, 0\rangle$$

(1pt.) $$\Rightarrow \boxed{2\sqrt{3}(x-\sqrt{3}) - 2(y-1) = 0}$$

$$\Rightarrow 2\sqrt{3}x - 6 - 2y + 2 = 0$$

$$\boxed{\sqrt{3}x - y = 2}$$

Solution 2 : Using level surface $f=4$

for $f(x,y,z) = x^2+(y-2)^2$ (1pt.)

$$\nabla f = \langle 2x,\ 2(y-2),\ 0\rangle \quad \text{(2pts.)}$$

$$\nabla f\Big|_{(\sqrt{3},1,1)} = \langle 2\sqrt{3}, -2, 0\rangle \quad \text{(2pts.)}$$

$$\Rightarrow \boxed{2\sqrt{3}(x-\sqrt{3}) - 2(y-1) = 0} \quad \text{(1pt.)}$$

7. Find the following limits:

a). $\lim_{(x,y)\to(0,0)} \frac{4e^y \sin(4x)}{5x} = \lim_{(x,y)\to(0,0)} \frac{4\cdot 4e^y \sin(4x)}{5\cdot 4x} = \frac{16}{5}$ (3pts.)

b). $\lim_{(x,y)\to(3,1)} \frac{xy-3y-5x+15}{x-3}. = -4$ (4pts.)

$\frac{\cancel{(x-3)}(y-5)}{\cancel{x-3}}$

c). Show that the limit does not exist:

$$\lim_{(x,y)\to(0,0)} \frac{y^4-2x^2}{y^4+x^2}.$$

Limit along parabolas:

$$\lim_{\substack{(x,y)\to(0,0)\\ x=ky^2}} \frac{y^4-2k^2y^4}{y^4+k^2y^2} = \frac{1-2k^2}{1+k^2}$$ (3pts.)

=> Limit DNE by Two-Path Test.

8. Consider the curve:

$$\mathbf{r}(t) = \langle 4\cos t,\ 4\sin t,\ 3t\rangle\,,$$

and its unit tangent and unit normal vectors:

$$\mathbf{T} = \left\langle -\frac{4}{5}\sin t,\ \frac{4}{5}\cos t,\ \frac{3}{5}\right\rangle,$$

$$\mathbf{N} = \langle -\cos t,\ -\sin t,\ 0\rangle\,.$$

a). Find the unit binormal vector **B**.

$$\vec{B} = \vec{T}\times\vec{N} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -4\sin t & 4\cos t & 3 \\ -\cos t & -\sin t & 0 \end{vmatrix}\frac{1}{5}$$

(5pts.)

$$= \frac{1}{5}\left\langle 3\sin t,\ -3\cos t,\ 4\right\rangle$$

b). Find the torsion τ along this curve.

$\vec{v} = \langle -4\sin t,\ 4\cos t,\ 3\rangle$ (1pt.)

$|\vec{v}| = \sqrt{16+9} = 5$ (1pt.)

$\dfrac{d\vec{B}}{dt} = \left\langle \frac{3}{5}\cos t,\ +\frac{3}{5}\sin t,\ 0\right\rangle$ (1pt.)

$\Rightarrow \tau = -\dfrac{d\vec{B}}{dt}\dfrac{1}{|\vec{v}|}\cdot\vec{N} = -\dfrac{1}{5}\cdot\dfrac{1}{5}\left(-3\cos^2 t - 3\sin^2 t\right) = 3/25$ (1pt.)

(1pt.)

$\boxed{9.}$ (a). Write parametric equations for the line joining the points $(0, 2, 0)$ and $(2, 0, 0)$.

Vector parallel to the line: $\langle 2, -2, 0\rangle$ (1pt.)

Parametric Equations: $\begin{cases} x = 2t \\ y = 2 - 2t \\ z = 0 \end{cases}$ (3pts.)

(b). Let C be the line segment from $(0, 2, 0)$ to $(2, 0, 0)$. Compute the line integral:

$$\int_C (x + y)\, ds.$$

$\vec{r}(t) = \langle 2t, 2-2t, 0\rangle$; $\underline{0 \le t \le 1}$ (1pt.)

$\vec{v}(t) = \langle 2, -2, 0\rangle$ (1pt.)

$|\vec{v}(t)| = 2\sqrt{2}$ (1pt.)

$f(\vec{r}(t)) = 2t + (2-2t) = 2$ (1pt.)

$\int_C f\, ds = \int_a^b f(\vec{r}(t))\, |\vec{v}(t)|\, dt$ (1pt.)

$= \int_0^1 2 \cdot 2\sqrt{2}\, dt = \boxed{4\sqrt{2}}$ (1pt.)

10. Consider the lines:

$$\text{L1:} \quad x = -1 + 3t, \quad y = 2 + 3t, \quad z = 1 - t;$$

$$\text{L2:} \quad x = 1 - 4s, \quad y = 1 + 2s, \quad z = 2 - 2s.$$

(a). Find the point of intersection of these lines.

$-1+3t = 1-4s \Rightarrow 4s+3t = 2$ $4s-3+6s=2 \Rightarrow s=\frac{1}{2}$ $t=0$

$1-t = 2-2s \Rightarrow t = -1+2s$

$(x,y,z) = (-1,2,1)$

(4pts.)

(b). Find an equation of the plane determined by the two lines.

$\vec{V_1} = \langle 3, 3, -1 \rangle$

$\vec{V_2} = \langle -4, 2, -2 \rangle$

(1pt.)

$$\vec{V_1} \times \vec{V_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 3 & -1 \\ -4 & 2 & -2 \end{vmatrix} = \langle -4, 10, 18 \rangle$$

(3pts.)

Plane : $\boxed{-4(x+1) + 10(y-2) + 18(z-1) = 0}$

(2pts.)

$-2x - 2 + 5y - 10 + 9z - 9 = 0$

$\boxed{-2x + 5y + 9z = 21}$

$\boxed{11.}$ Consider the curve:

$$\mathbf{r}(t) = \langle -\sin t,\ \cos t,\ 2t\rangle .$$

(a). Find the arc length parameter along this curve, taking $(0,1,0)$ for the initial point.

$t=0$

$$\vec{v}(t) = \langle -\cos t, -\sin t, 2\rangle$$

$$|\vec{v}(t)| = \sqrt{1+4} = \sqrt{5}$$

$$s(t) = \int_0^t |\vec{v}(\tau)|\,d\tau = \sqrt{5}\,t$$

(4pts.)

(b). Find the length of the portion of this curve with $0 \le t \le \pi/6$.

$$s(\pi/6) = \frac{\sqrt{5}\,\pi}{6}$$

(3pts.)

(c). Find the point on this curve that is at distance $\sqrt{5}\pi/2$ units along the curve from $(0,1,0)$ in the direction of increasing arc length.

$$s(t) = \frac{\sqrt{5}\,\pi}{2}$$

$$\sqrt{5}\,t = \sqrt{5}\,\pi/2 \Rightarrow t = \pi/2 \Rightarrow P = (-1, 0, \pi)$$

(3pts.)

$\boxed{12.}$ Given that for a curve $\mathbf{r}(t)$:

$$\frac{d\mathbf{r}}{dt} = t(e^{t^2} + 1)\,\mathbf{i} + \frac{e^t}{e^t + 1}\,\mathbf{j} + 0\,\mathbf{k},$$

and that:

$$\mathbf{r}(0) = \langle 1, 0, 2 \rangle\,,$$

find $\mathbf{r}(t)$.

$$\vec{r}(t) = \left\langle \frac{1}{2}e^{t^2} + \frac{t^2}{2} + c_1,\ \ln(e^t+1) + c_2,\ c_3 \right\rangle$$

(6pts.)

+ln(2)

$$\Rightarrow \vec{r}(0) = \left\langle \frac{1}{2} + c_1,\ c_2,\ c_3 \right\rangle$$

$$= \langle 1, 0, 2 \rangle \quad \Rightarrow \quad \begin{array}{l} c_1 = 1/2 \\ c_2 = 0 \ -\ln(2) \\ c_3 = 2 \end{array}$$

(3pts.)

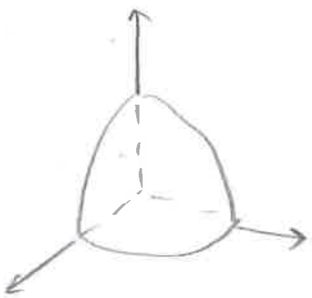
$$\Rightarrow \boxed{\vec{r}(t) = \left\langle \frac{1}{2}e^{t^2} + \frac{t^2}{2} + \frac{1}{2},\ \ln(e^t+1) - \ln(2),\ 2 \right\rangle}$$

(1pt.)

$\boxed{13.}$ Find the outward flux of the field:

$$\mathbf{F}(x,y,z) = \langle x^4,\ -4x^3y,\ 2xz \rangle,$$

across the boundary of the region D cut from the first octant (where x, y, z are all positive) by the sphere $x^2+y^2+z^2=4$.



$$\nabla\cdot\vec{F} = M_x + N_y + P_z$$
$$= 4x^3 - 4x^3 + 2x$$
$$= 2x \qquad \text{(5pts.)}$$
$$= 2\rho \sin\phi\cos\theta$$

Divergence Theorem :

$$\iint_S \vec{F}\cdot\vec{n}\,d\sigma = \iiint_D \nabla\cdot\vec{F}\,dV \qquad \text{(1pt.)}$$
$$= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^2 2\rho^3\sin^2\phi\cos\theta\,d\rho\,d\phi\,d\theta$$
$$= \left(\frac{2\rho^4}{4}\Big|_0^2\right)\left(\frac{1}{2}\phi - \frac{1}{4}\sin(2\phi)\right)\Big|_0^{\pi/2}\left(\sin\theta\Big|_0^{\pi/2}\right)$$
$$= (8)\left(\frac{\pi}{4}\right)$$
$$= \boxed{2\pi} \qquad \text{(4pts.)}$$

$\boxed{14.}$ Prove that

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}.$$

$$I = \int_0^\infty e^{-x^2} dx$$

$$I^2 = \int_0^\infty e^{-x^2} dx \int_0^\infty e^{-y^2} dy \qquad \text{(2pts.)}$$

$$= \int_0^\infty \int_0^\infty e^{-x^2-y^2} dx\,dy \qquad \text{(1pt.)}$$

$$= \int_0^{\pi/2} \int_0^\infty e^{-r^2} r\,dr\,d\theta \qquad \text{(4pts.)}$$

$$= \frac{\pi}{2}\left(-\frac{1}{2}e^{-r^2}\right)\Big|_0^\infty$$

$$= \frac{\pi}{2}\cdot\frac{1}{2}$$

$$= \pi/4 \qquad \text{(3pts.)}$$

$$\Rightarrow I = \frac{\sqrt{\pi}}{2}$$