Name: Solutions

May 5th, 2016.
Math 2551; Sections L1, L2, L3.
Georgia Institute of Technology
Final Exam

I commit to uphold the ideals of honor and integrity by refusing to betray the trust bestowed upon me as a member of the Georgia Tech community. By signing my name below I pledge that I have neither given nor received help on this exam.

Pledged: ______________________

Problem	Possible Score	Earned Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
11	10	
12	10	
13	10	
14	10	
Total	140	

Remember that you must SHOW YOUR WORK to receive credit!

Good luck!

$\boxed{1.}$ (a). Write parametric equations for the line joining the points $(0,8,0)$ and $(8,0,0)$.

Vector parallel to the line: $\langle 8,-8,0\rangle$ (1pt.)

Parametric Equations: $\begin{cases} x = 8t \\ y = 8-8t \\ z = 0 \end{cases}$ (3pts.)

(b). Let C be the line segment from $(0,8,0)$ to $(8,0,0)$. Compute the line integral:

$$\int_C (x+y)\,ds.$$

$\int_C f\,ds = \int_a^b f(\vec{r}(t))|\vec{v}(t)|\,dt$ (1pt.)

$\vec{r}(t) = \langle 8t,\ 8-8t,\ 0\rangle$; $\underline{0 \leq t \leq 1}$ (1 pt.)

$f(x,y,z) = x+y$

$f(\vec{r}(t)) = (8t) + (8-8t) = 8$ (1pt.)

$\vec{v}(t) = \langle 8,-8,0\rangle$ (1pt.)

$|\vec{v}(t)| = 8\sqrt{2}$ (1pt.)

$\Rightarrow \int_C (x+y)\,ds = \int_0^1 8\cdot 8\sqrt{2}\,dt = 64\sqrt{2}$ (1pt.)

$\boxed{2.}$ Given that for a curve $\mathbf{r}(t)$:

$$\frac{d\mathbf{r}}{dt} = \frac{1}{\sqrt{t}}\,\mathbf{i} + \sin(t)\,e^{\cos(t)}\,\mathbf{j} + t\,\mathbf{k},$$

and that:

$$\mathbf{r}(0) = \langle 1, 0, 3 \rangle\,,$$

find $\mathbf{r}(t)$.

$$\vec{r}(t) = \left\langle 2\sqrt{t} + C_1,\ -e^{\cos(t)} + C_2,\ \frac{t^2}{2} + C_3 \right\rangle$$

(6pts.)

$$\vec{r}(0) = \langle C_1,\ -e + C_2,\ C_3 \rangle$$

$$= \langle 1,\ 0,\ 3 \rangle$$

$\Rightarrow C_1 = 1$
$C_2 = e$
$C_3 = 3$

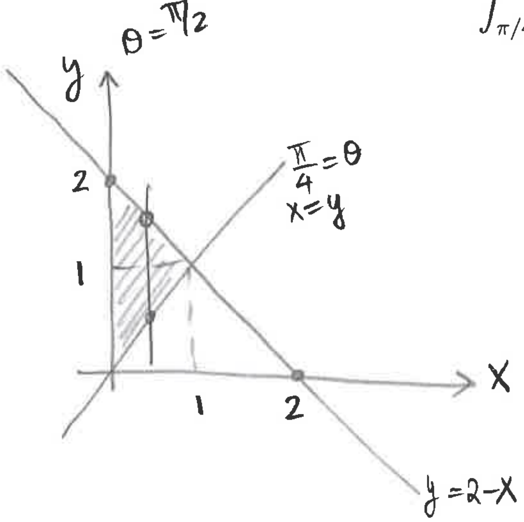
(3pts.)

$$\Rightarrow \boxed{\vec{r}(t) = \left\langle 2\sqrt{t} + 1,\ -e^{\cos(t)} + e,\ \frac{t^2}{2} + 3 \right\rangle}$$

(1pt.)

$\boxed{3.}$ Compute

$$\int_{\pi/4}^{\pi/2} \int_0^{\frac{2}{\cos\theta+\sin\theta}} r^2 \cos\theta \, dr \, d\theta.$$



$$r = \frac{2}{\cos\theta + \sin\theta}$$

$$r\cos\theta + r\sin\theta = 2$$

$$x + y = 2$$

$$y = 2 - x$$

(6pts).

Convert to Cartesian:

$$\int_0^1 \int_x^{2-x} x \, dy \, dx$$ (3pts.)

$$= \int_0^1 x(2-x-x)\,dx$$

$$= \int_0^1 (2x - 2x^2)\,dx$$

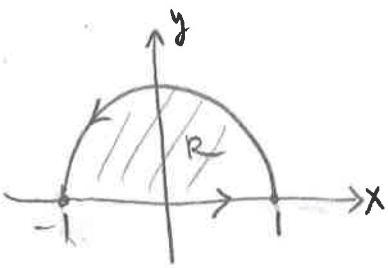
$$= \left(x^2 - \frac{2x^3}{3}\right)\Big|_0^1 = 1/3$$ (1pt.)

$\boxed{4.}$ (a). Find

(5pts.)

$$\oint_C y^2\,dx + 3xy\,dy,$$

where C is the closed, positively oriented curve consisting of the upper half of the unit circle $x^2+y^2=1$ ($y>0$), and the line segment joining $(-1,0)$ and $(1,0)$.



Green : $\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$

$M = y^2$

$N = 3xy$

$$= \iint_R (3y - 2y)\,dA = \iint_R y\,dA$$

$$= \int_0^{\pi}\int_0^1 r^2 \sin\theta\, dr\, d\theta$$

$$= (-\cos\theta)\Big|_0^{\pi} \left(\frac{r^3}{3}\right)\Big|_0^1$$

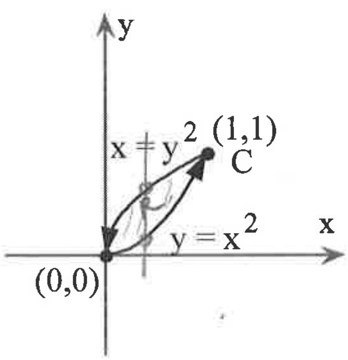
$$= 2/3$$

(b). Find the outward flux of the field

(5pts)

$$\mathbf{F}(x,y) = \langle 2xy + y^2,\ 2x - y\rangle,$$

across the curve C pictured below.



Green : $\oint_C \vec{F}\cdot\vec{n}\,ds = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA$

$M = 2xy + y^2$

$N = 2x - y$

$$= \iint_R (2y - 1)\,dA$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} (2y-1)\,dy\,dx$$

$$= \int_0^1 (y^2 - y)\Big|_{x^2}^{\sqrt{x}}\,dx$$

$$= \int_0^1 (x - \sqrt{x}) - (x^4 - x^2)\,dx$$

$$= \left(\frac{x^2}{2} - \frac{2}{3}x^{3/2} - \frac{x^5}{5} + \frac{x^3}{3}\right)\Big|_0^1$$

$$= \frac{1}{2} - \frac{2}{3} - \frac{1}{5} + \frac{1}{3}$$

$$= \frac{3}{10} - \frac{1}{3}$$

$$= -1/30$$

5. Find the following limits:

a). $\lim\limits_{(x,y)\to(0,0)} \dfrac{e^{-2y}\sin(3x)}{5x} = \lim\limits_{(x,y)\to(0,0)} \dfrac{3e^{-2y}\,\overbrace{\sin(3x)}^{\nearrow 1}}{5\cdot 3x} = \dfrac{3}{5}$ (3pts.)

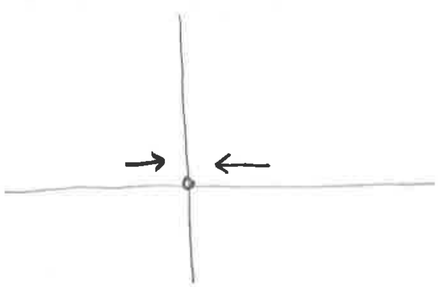
b). $\lim\limits_{(x,y)\to(2,8)} \dfrac{xy-2y-7x+14}{x-2} = 1$ (4pts.)

$$\frac{\cancel{(x-2)}(y-7)}{\cancel{x-2}}$$

c). Show that the limit does not exist:

$$\lim_{(x,y)\to(0,0)} \frac{x}{\sqrt{x^2+y^2}}.$$

(3pts.)



Limit along positive x-axis:

$$\lim_{\substack{x\to 0\\ x>0}} \frac{x}{\sqrt{x^2}} = \lim_{\substack{x\to 0\\ x>0}} \frac{x}{x} = \boxed{1}$$

Limit along negative x-axis:

$$\lim_{\substack{x\to 0\\ x<0}} \frac{x}{\sqrt{x^2}} = \lim_{\substack{x\to 0\\ x<0}} \frac{x}{-x} = \boxed{-1}$$

=> Limit DNE by the Two-Path Test.

6. Find the minimum and maximum value of $f(x,y) = x^2 + y^2$ subject to the constraint

$$x^2 - 4x + y^2 - 2y = 0.$$

Indicate the points where these extreme values occur.

Lagrange Multipliers:

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 0 \end{cases}$$

(2pts.) Setup

$f(x,y) = x^2 + y^2$

$g(x,y) = x^2 - 4x + y^2 - 2y = 0$

$\nabla f = \langle 2x, 2y \rangle; \quad \nabla g = \langle 2x-4, 2y-2 \rangle.$ (2pts.)

$$\begin{cases} 2x = \lambda(2x-4) \longrightarrow x = \lambda(x-2) \longrightarrow (\lambda-1)x = 2\lambda \Rightarrow x = \dfrac{2\lambda}{\lambda-1} \\ 2y = \lambda(2y-2) \longrightarrow y = \lambda(y-1) \longrightarrow (\lambda-1)y = \lambda \Rightarrow y = \dfrac{\lambda}{\lambda-1} \\ x^2 - 4x + y^2 - 2y = 0 \end{cases}$$

$x = 2y$

$4y^2 - 8y + y^2 - 2y = 0$

$5y^2 - 10y = 0$

$5y(y-2) = 0 \Rightarrow y = 0$ or $y = 2$

$\Rightarrow$ Extreme values occur at $(0,0)$ and $(4,2)$

$f(0,0) = 0 \Rightarrow$ $\boxed{\text{minimum value} = 0, \text{ at } (0,0).}$ (3pts.)

$f(4,2) = 16 + 4 = 20 \Rightarrow$ $\boxed{\text{maximum value} = 20, \text{ at } (4,2)}$ (3pts.)

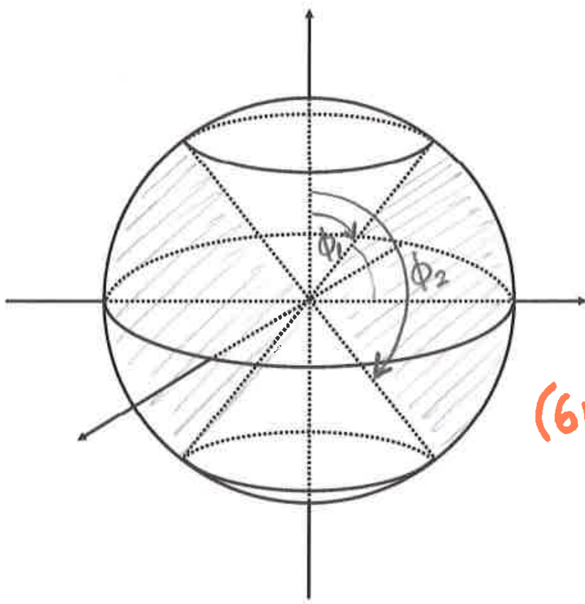
((2pts.) for each point
(1pt.) for value)

7. (a). Let x be a real number. What is $\sqrt{x^2}$?

$\sqrt{x^2} = |X|$

(2pts.)

(b). Let D be the solid that lies *inside* the sphere $x^2+y^2+z^2=4$ and *outside* the cone $z^2=3(x^2+y^2)$. Use *spherical* coordinates to set up the triple integral that gives the volume of D. You do not need to compute the volume.



(6pts.)

Find ϕ_1, ϕ_2 (angles of the cone)

$z^2 = 3r^2$ (Cone)

$\left.\begin{aligned} z &= \rho\cos\phi \\ r &= \rho\sin\phi \end{aligned}\right\}$ in spherical $\Rightarrow$

$\Rightarrow \cancel{\rho^2}\cos^2\phi = 3\cancel{\rho^2}\sin^2\phi$

$\Rightarrow \tan^2\phi = \frac{1}{3}$

$\Rightarrow \tan\phi = \pm\frac{1}{\sqrt{3}}$

$\Rightarrow \phi_1 = \frac{\pi}{6}$ $\quad \phi_2 = \frac{5\pi}{6}$

$$V = \int_0^{2\pi}\int_{\pi/6}^{5\pi/6}\int_0^2 \rho^2\sin\phi \, d\rho\, d\phi\, d\theta$$

(2pts.)

8. Find the outward flux of the field:

$$\mathbf{F}(x,y,z) = \langle 3x^3 + 3xy^2,\ 2y^3 + e^y \sin z,\ 3z^3 + e^y \cos z \rangle,$$

across the boundary of the region D given by $1 \le x^2 + y^2 + z^2 \le 2$.

$$\nabla \cdot \vec{F} = M_x + N_y + P_z$$
$$= (9x^2 + 3y^2) + (6y^2 + \cancel{e^y \sin z}) + (9z^2 - \cancel{e^y \sin z})$$
$$= 9(x^2 + y^2 + z^2)$$

(5pts.)

Divergence Theorem :

$$\iint_S \vec{F} \cdot \vec{n}\, d\sigma = \iiint_D \nabla \cdot \vec{F}\, dV$$

(1pt.)

$$= \int_0^{2\pi} \int_0^{\pi} \int_1^{\sqrt{2}} (9\rho^2)\rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$
$$= \left(\frac{9\rho^5}{5} \Big|_1^{\sqrt{2}} \right) (2\pi) (-\cos\phi)\Big|_0^{\pi}$$
$$= 4\pi \left(\frac{9}{5}\right)(4\sqrt{2} - 1)$$
$$= \boxed{\frac{36\pi}{5}(4\sqrt{2} - 1)}$$

(4pts.)

9. Consider the curve:

$$\mathbf{r}(t) = \left\langle 2\cos t,\ 2\sin t,\ \sqrt{5}t \right\rangle.$$

(a). Find the arc length parameter along this curve, taking $\underbrace{(2,0,0)}_{t=0}$ for the initial point.

(4pts.)

$$s(t) = \int_0^t |\vec{v}(\tau)|\,d\tau$$

$$= \int_0^t 3\,d\tau$$

$$\Rightarrow \boxed{s(t) = 3t}$$

$\vec{v}(t) = \langle -2\sin t, 2\cos t, \sqrt{5} \rangle$

$|\vec{v}(t)| = \sqrt{4+5} = 3$

(b). Find the length of the portion of this curve with $0 \le t \le \pi/6$.

(3pts.)

$$s(\pi/6) = \pi/2$$

(c). Find the point on this curve that is at distance π units along the curve from $(2,0,0)$ in the direction of increasing arc length.

(3pts.)

$$s(t) = \pi \Rightarrow 3t = \pi \Rightarrow t = \pi/3$$

$$\Rightarrow P_0 = \left(2\cos \pi/3,\ 2\sin \pi/3,\ \sqrt{5}\,\pi/3\right)$$

$$= \boxed{\left(1, \sqrt{3}, \frac{\sqrt{5}\pi}{3}\right)}$$

10. Consider the surface S given by the parametrization:

$$\mathbf{r}(r,\theta) = \langle r\cos\theta,\ r\sin\theta,\ r\rangle;\ r \ge 0;\ 0 \le \theta \le 2\pi.$$

(a). Find a Cartesian equation for the surface S and sketch it.

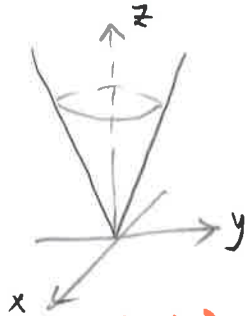
$$\left.\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right\} \Rightarrow r = \sqrt{x^2+y^2}$$

$$z = r$$

$$z = \sqrt{x^2+y^2}$$

↳ cone opening up from the origin

(3pts.)



(1pt.)

(b). Find the equation of the plane tangent to the surface S at the point $P_0(-1, \sqrt{3}, 2)$ corresponding to $(r,\theta) = (2, 2\pi/3)$.

Solution 1 : Using the parametrization :

(2pts.)

$$\vec{r}_r = \langle \cos\theta,\ \sin\theta,\ 1 \rangle$$

$$\vec{r}_\theta = \langle -r\sin\theta,\ r\cos\theta,\ 0 \rangle$$

(1pt) $$\vec{r}_r \times \vec{r}_\theta = \langle -r\cos\theta,\ -r\sin\theta,\ r \rangle$$

$$\vec{r}_r \times \vec{r}_\theta \Big|_{(2, 2\pi/3)} = \left\langle -2\left(-\tfrac{1}{2}\right),\ -2\left(\tfrac{\sqrt{3}}{2}\right),\ 2 \right\rangle$$

(2pts.)

$$= \langle 1, -\sqrt{3}, 2 \rangle$$

↳ Vector perp. to plane

$\Rightarrow$ Plane equation:

(1pt.) $$\boxed{1(x+1) - \sqrt{3}(y-\sqrt{3}) + 2(z-2) = 0}$$

$$\Rightarrow \boxed{x - \sqrt{3}y + 2z = 0}$$

Solution 2 : Using the gradient

Surface : $z = \sqrt{x^2+y^2}$

$\Rightarrow$ Level surface $f = 0$ for

$$f(x,y,z) = \sqrt{x^2+y^2} - z$$ (1pt.)

$$\nabla f = \left\langle \frac{x}{\sqrt{x^2+y^2}},\ \frac{y}{\sqrt{x^2+y^2}},\ -1 \right\rangle$$ (2pts.)

Evaluate $\nabla f \big|_{(-1,\sqrt{3},2)}$

$$\left\langle \frac{-1}{2},\ \frac{\sqrt{3}}{2},\ -1 \right\rangle$$ (2pts.)

$\Rightarrow$ Plane Equation:

$$\boxed{-\tfrac{1}{2}(x+1) + \tfrac{\sqrt{3}}{2}(y-\sqrt{3}) - 1(z-2) = 0}$$ (1pt)

$$x + 1 - \sqrt{3}(y-\sqrt{3}) + 2(z-2) = 0$$

$$x - \sqrt{3}y + 2z = 0$$

11. Find all the critical points of the function:

$$f(x,y) = x^3 + y^3 + 3x^2 - 6y^2 - 1,$$

and classify each one as a local minimum, a local maximum, or a saddle point.

(1pt.) $f_x = 3x^2 + 6x$

$f_y = 3y^2 - 12y$

(1pt.) $\begin{cases} 3x(x+2)=0 \\ 3y(y-4)=0 \end{cases}$ $\quad \begin{cases} x=0 \text{ or } x=-2 \\ y=0 \text{ or } y=4 \end{cases}$

$\Rightarrow$ Critical points: $\boxed{(0,0)}$, $\boxed{(0,4)}$, $\boxed{(-2,0)}$, $\boxed{(-2,4)}$ (4pts.)

$f_{xx} = 6x+6$

$f_{yy} = 6y-12$

$f_{xy} = 0$

$\Delta_f = f_{xx} f_{yy} - f_{xy}^2 = 36(x+1)(y-2)$

$\Delta_f(0,0) < 0 \Rightarrow$ Saddle point $(0,0)$

$\left.\begin{array}{l} \Delta_f(0,4) > 0 \\ f_{xx}(0,4) > 0 \end{array}\right\} \Rightarrow$ local min $(0,4)$ (4pts.)

$\left.\begin{array}{l} \Delta_f(-2,0) > 0 \\ f_{xx}(-2,0) < 0 \end{array}\right\} \Rightarrow$ local max $(-2,0)$

$\Delta_f(-2,4) < 0 \Rightarrow$ Saddle point $(-2,4)$

12. Consider the curve:

$$\mathbf{r}(t) = \langle 6\sin t,\ 6\cos t,\ 8t\rangle,$$

and its unit tangent and unit normal vectors:

$$\mathbf{T} = \left\langle \frac{3}{5}\cos t,\ -\frac{3}{5}\sin t,\ \frac{4}{5}\right\rangle,$$

$$\mathbf{N} = \langle -\sin t,\ -\cos t,\ 0\rangle.$$

a). Find the unit binormal vector **B**.

$$\vec{B} = \vec{T}\times\vec{N} = \frac{1}{5}\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3\cos t & -3\sin t & 4 \\ -\sin t & -\cos t & 0 \end{vmatrix}$$

$$= \frac{1}{5}\left\langle 4\cos t,\ -4\sin t,\ -3\right\rangle$$

(5pts.)

b). Find the torsion τ along this curve.

$$\frac{d\vec{B}}{dt} = \frac{1}{5}\left\langle -4\sin t,\ -4\cos t,\ 0\right\rangle$$ (1pt.)

$$\vec{v} = \left\langle 6\cos t,\ -6\sin t,\ 8\right\rangle$$ (1pt.)

$$|\vec{v}| = \sqrt{36+64} = 10$$ (1pt.)

$$\Rightarrow \tau = -\frac{d\vec{B}}{dt}\frac{1}{|\vec{v}|}\cdot\vec{N} = \frac{-1}{50}\left(4\sin^2 t + 4\cos^2 t\right) = \boxed{\frac{-4}{50}} = \boxed{\frac{-2}{25}}$$ (1pt.)

(1pt.)

13. Consider the lines:

$$\text{L1:} \quad x = -1 + 2t, \quad y = 2 + 3t, \quad z = 1 - 2t;$$

$$\text{L2:} \quad x = 1 - 4s, \quad y = 1 + 2s, \quad z = 2 - 2s.$$

(a). Find the point of intersection of these lines.

$-1+2t = 1-4s \Rightarrow 2s = 1-t$

$2+3t = 1+2s$

$= 2-t \Rightarrow t=0 \Rightarrow s=1/2 \Rightarrow (x,y,z)=(-1,2,1)$ (4pts.)

(b). Find an equation of the plane determined by the two lines.

Vector parallel to L_1: $\vec{v}_1 = \langle 2,3,-2\rangle$ (1pt.)

L_2: $\vec{v}_2 = \langle -4,2,-2\rangle$

$$\vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & -2 \\ -4 & 2 & -2 \end{vmatrix} = \langle -2, 12, 16\rangle$$ (3pts.)

Plane: $\boxed{-2(x+1) + 12(y-2) + 16(z-1) = 0}$ (2pts.)

$-x-1+6y-12+8z-8=0$

$\boxed{-x+6y+8z=21}$

14. Prove that:

$$\int_{-\infty}^{\infty} e^{-ax^2}\, dx = \sqrt{\frac{\pi}{a}},$$

where a is any positive real number.

$$I = \int_{-\infty}^{\infty} e^{-ax^2} dx$$

$$I^2 = \int_{-\infty}^{\infty} e^{-ax^2} dx \int_{-\infty}^{\infty} e^{-ay^2} dy$$ (3pts.)

(2pts.)

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-a(x^2+y^2)} dx\,dy$$

(3pts.)

$$= \int_{0}^{2\pi}\int_{0}^{\infty} e^{-ar^2} r\,dr\,d\theta$$

$$= 2\pi \left(\frac{-1}{2a} e^{-ar^2} \Big|_0^{\infty} \right)$$

$$= 2\pi \left(0 - \frac{-1}{2a} \right)$$

(2pts.)

$$= \pi/a$$

$$\Rightarrow I = \sqrt{\pi/a}$$