Name: Solutions

October 1st, 2014.
Math 2401; Sections D1, D2, D3.
Georgia Institute of Technology
Exam 2

I commit to uphold the ideals of honor and integrity by refusing to betray the trust bestowed upon me as a member of the Georgia Tech community. By signing my name below I pledge that I have neither given nor received help on this exam.

Pledged: ____________________

Problem	Possible Score	Earned Score
1	20	
2	15	
3	15	
4	15	
5	20	
6	15	
Total	100	

Remember that you must SHOW YOUR WORK to receive credit!

Good luck!

2. [15 points] Given that y is defined implicitly in terms of x by:

$$y = \sin(3x + 4y),$$

find $\frac{dy}{dx}$.

Method I : Partial Derivatives

$F(x,y) = y - \sin(3x+4y) = 0$ 3.5 pts

$$\underbrace{\frac{dy}{dx} = -\frac{F_x}{F_y}}_{3.5\text{ pts.}} = -\frac{-\cos(3x+4y)\cdot 3}{1-\cos(3x+4y)\cdot 4}$$

$$= \frac{3\cos(3x+4y)}{1-4\cos(3x+4y)}$$

Correct $F_x \rightarrow$ 4.5 pts.

Correct $F_y \rightarrow$ 4.5 pts.

Method II : Implicit Differentiation

3 pts. $\frac{dy}{dx} = \frac{d}{dx}\sin(3x+4y)$

3 pts. $\frac{dy}{dx} = \cos(3x+4y)\left(3+4\frac{dy}{dx}\right)$

3 pts. $\frac{dy}{dx} = 3\cos(3x+4y) + 4\cos(3x+4y)\cdot\frac{dy}{dx}$

3 pts. $\left(1-4\cos(3x+4y)\right)\frac{dy}{dx} = 3\cos(3x+4y)$

3 pts. $\frac{dy}{dx} = \frac{3\cos(3x+4y)}{1-4\cos(3x+4y)}$

3. [15 points] Determine whether or not the function $f(x, y) = e^{-2y}\cos(2x)$ satisfies the two-dimensional Laplace equation:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

3.5 pts. $\frac{\partial f}{\partial x} = -2e^{-2y}\sin(2x)$

$\frac{\partial f}{\partial y} = -2e^{-2y}\cos(2x)$ 3.5 pts.

3.5 pts. $\frac{\partial^2 f}{\partial x^2} = -4e^{-2y}\cos(2x)$

$\frac{\partial^2 f}{\partial y^2} = 4e^{-2y}\cos(2x)$ 3.5 pts.

$\Rightarrow \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$ is true 1 pt.

4. [15 points] Consider the function:

$$f(x,y) = \ln(x^2 + y^4).$$

[6pts.] a. Find the gradient of f.

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \left\langle \frac{2x}{x^2+y^4}, \frac{4y^3}{x^2+y^4} \right\rangle$$

3pts. 3pts.

[9pts.] b. Find the directions $\vec{u}$ where $D_{\vec{u}}f(P_0) = 0$, where $P_0(1,1)$.

$$(\nabla f)_{P_0} = \left\langle \frac{2\cdot 1}{1+1}, \frac{4\cdot 1}{1+1} \right\rangle = \langle 1, 2 \rangle$$

2pts. - evaluating ∇f at P_0

$$D_{\vec{u}}f(P_0) = (\nabla f)_{P_0} \cdot \vec{u}$$

2pts. - using correct formula for $D_{\vec{u}}f$

$$\vec{u} = \langle u_1, u_2 \rangle$$

2pts. - setting up correct system of equations

$$(\nabla f)_{P_0} \cdot \vec{u} = u_1 + 2u_2$$

2pts. - Solving the system of equations

$$\begin{cases} u_1 + 2u_2 = 0 \\ u_1^2 + u_2^2 = 1 \end{cases} \qquad \begin{cases} u_1 = -2u_2 \\ u_1^2 + u_2^2 = 1 \end{cases}$$

$$(-2u_2)^2 + u_2^2 = 1$$

$$4u_2^2 + u_2^2 = 1$$

$$5u_2^2 = 1$$

$$u_2^2 = \frac{1}{5}$$

$$u_2 = \pm\frac{1}{\sqrt{5}}$$

$$u_2 = \frac{1}{\sqrt{5}} \Rightarrow u_1 = -\frac{2}{\sqrt{5}}$$

$$u_2 = \frac{-1}{\sqrt{5}} \Rightarrow u_1 = \frac{2}{\sqrt{5}}$$

Directions: $\boxed{\left(\frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right), \left(\frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\right)}$

1pt. - final answer

5. [20 points] Consider the function:

$$f(x,y) = x^3 + y^3 + 6x^2 - 3y^2 - 5.$$

[10pts.] a. Find the critical points of f.

$f_x = 3x^2 + 12x$ 2pts.

$f_y = 3y^2 - 6y$ 2pts.

$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases}$ 2pts. setting up system of equations $\begin{cases} 3x^2+12x=0 \\ 3y^2-6y=0 \end{cases}$ $\begin{cases} 3x(x+4)=0 \\ 3y(y-2)=0 \end{cases}$ $\begin{cases} x=0,-4 \\ y=0,2 \end{cases}$ 2pts. Solving system

Critical Points : $(0,0)$, $(0,2)$, $(-4,0)$, $(-4,2)$

½ pt. ½ pt. ½ pt. ½ pt.

[10pts.] b. Use the Second Derivative Test to classify each critical point as a saddle point, a local minimum, or a local maximum.

$f_{xx} = 6x + 12$ 1pt.

$f_{yy} = 6y - 6$ 1pt.

$f_{xy} = 0$ 1pt.

$f_{xx}f_{yy} - f_{xy}^2 = (6x+12)(6y-6)$ 1pt.

$\boxed{(0,0)} \to \left(f_{xx}f_{yy} - f_{xy}^2\right)\Big|_{(0,0)} = 12(-6) < 0$ saddle point 1pt.

$\boxed{(0,2)} \to \left(f_{xx}f_{yy} - f_{xy}^2\right)\Big|_{(0,2)} = 12(12-6) > 0$ 1pt.

$f_{xx}\Big|_{(0,2)} = 0 + 12 > 0$ local min 1pt.

$\boxed{(-4,0)} \to \left(f_{xx}f_{yy} - f_{xy}^2\right)\Big|_{(-4,0)} = (-24+12)(-6) > 0$ 1pt.

$f_{xx}\Big|_{(-4,0)} = -24 + 12 < 0$ local max 1pt.

$\boxed{(-4,2)} \to \left(f_{xx}f_{yy} - f_{xy}^2\right)\Big|_{(-4,2)} = (-24+12)(12-6) < 0$ saddle point 1pt.

6. [15 points] Find the point on the sphere $x^2 + y^2 + z^2 = 4$ that is farthest from the point $(-1, -1, -1)$.

Maximize: $f(x,y,z) = (x+1)^2 + (y+1)^2 + (z+1)^2$ 1pt.

Subject to: $g(x,y,z) = x^2+y^2+z^2-4 = 0$ 1pt.

$$\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y,z) = 0 \end{cases}$$ 1pt.

$\nabla f = \langle 2(x+1), 2(y+1), 2(z+1) \rangle$ 2pts.

$\nabla g = \langle 2x, 2y, 2z \rangle$ 2pts.

1pt. Setting up system

$$\begin{cases} 2(x+1) = 2\lambda x \\ 2(y+1) = 2\lambda y \\ 2(z+1) = 2\lambda z \\ x^2+y^2+z^2 = 4 \end{cases} \qquad \begin{cases} x+1 = \lambda x \\ y+1 = \lambda y \\ z+1 = \lambda z \\ x^2+y^2+z^2 = 4 \end{cases} \qquad \begin{cases} (1-\lambda)x = -1 \\ (1-\lambda)y = -1 \\ (1-\lambda)z = -1 \\ x^2+y^2+z^2 = 4 \end{cases}$$

$x = y = z = -\frac{1}{1-\lambda}$ 3pts. – solving system

$x^2+y^2+z^2=4$ becomes $x^2+x^2+x^2 = 4$

$3x^2 = 4$

$x^2 = \frac{4}{3}$

$x = \pm \frac{2}{\sqrt{3}}$

Possible Solutions: $\underbrace{\left(\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)}_{\text{maximum}}$, $\underbrace{\left(-\frac{2}{\sqrt{3}}, -\frac{2}{\sqrt{3}}, -\frac{2}{\sqrt{3}}\right)}_{\text{minimum}}$

maximum 1pt. minimum 1pt.

Answer: $\boxed{\left(\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)}$ 2pts. – choosing correct solution

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October 29$^{\text{th}}$, 2014.
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Exam 3

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4	20	
5	20	
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1. [20 points] Find the volume of the region in space bounded above by the surface $z = 4\cos(x)\sin(y)$ and below by the rectangle: $0 \le x \le \pi/6$, $0 \le y \le \pi/4$.

(2pts.) $V = \iint_R f(x,y)\,dA$

(6pts.) $= \int_0^{\pi/6}\int_0^{\pi/4} 4\cos(x)\sin(y)\,dy\,dx$ $\left(\text{or } \int_0^{\pi/4}\int_0^{\pi/6} 4\cos(x)\sin(y)\,dx\,dy\right)$

(2pts.) $= \int_0^{\pi/6} -4\cos(x)\cos(y)\Big|_{y=0}^{y=\pi/4}\,dx$ $\left(= \int_0^{\pi/4} 4\sin(x)\sin(y)\Big|_{x=0}^{x=\pi/6}\,dy\right)$

(2pts.) $= \int_0^{\pi/6}\left(-4\cos(x)\cdot\frac{\sqrt{2}}{2} + 4\cos(x)\right)dx$ $\left(= \int_0^{\pi/4} 4\cdot\frac{1}{2}\sin(y)\,dy\right)$

(2pts.) $= \int_0^{\pi/6}(4-2\sqrt{2})\cos(x)\,dx$ $\left(= \int_0^{\pi/4} 2\sin(y)\,dy\right)$

(2pts.) $= (4-2\sqrt{2})\sin(x)\Big|_{x=0}^{x=\pi/6}$ $\left(= -2\cos(y)\Big|_{y=0}^{y=\pi/4}\right)$

(2pts.) $= (4-2\sqrt{2})\cdot\frac{1}{2}$ $\left(= -2\cdot\frac{\sqrt{2}}{2} + 2\cdot 1\right)$

(2pts.) $= \boxed{2-\sqrt{2}}$ $\left(= \boxed{2-\sqrt{2}}\right)$

2 pts. Correct bounds for x

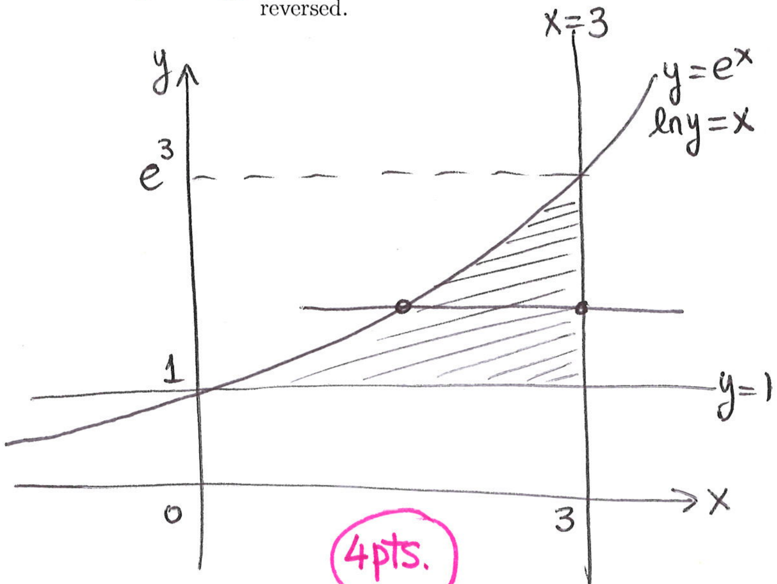
2pts. correct bounds for y

2pts. correct order of dx, dy.

2. [20 points] Given the integral below:

$$\int_0^3 \int_1^{e^x} \frac{1}{y}\, dy\, dx.$$

[12 pts.] a). Sketch the region of integration and write an equivalent double integral with the order of integration reversed.



4pts.

$1 \leq y \leq e^x$

$0 \leq x \leq 3$

Horizontal Cross-Sections:

$$\boxed{\int_1^{e^3} \int_{\ln(y)}^{3} \frac{1}{y}\, dx\, dy}$$

8pts. — 4pts. bounds for x; 4pts. bounds for y

[8 pts.] b). Use either one of the two versions of the integral to compute its value.

$$\int_0^3 \int_1^{e^x} \frac{1}{y}\, dy\, dx$$

$$= \int_0^3 \ln(y) \Big|_{y=1}^{y=e^x} dx \quad \text{2pts.}$$

$$= \int_0^3 \left(\ln(e^x) - \ln(1)\right) dx \quad \text{2pts.}$$

$$= \int_0^3 (x)\, dx \quad \text{2pts.}$$

$$= \frac{x^2}{2}\Big|_0^3 \quad \text{1pt.}$$

$$= \boxed{9/2} \quad \text{1pt.}$$

or

$$\int_1^{e^3} \int_{\ln(y)}^{3} \frac{1}{y}\, dx\, dy$$

$$= \int_1^{e^3} \frac{1}{y} x \Big|_{x=\ln(y)}^{x=3} dy \quad \text{2pts.}$$

$$= \int_1^{e^3} \left(\frac{3}{y} - \frac{1}{y}\ln(y)\right) dy \quad \text{2pts.}$$

$$= \left(3\ln(y) - \frac{\ln^2(y)}{2}\right)\Big|_{y=1}^{y=e^3} \quad \text{2pts.}$$

$$= 3\ln(e^3) - \frac{(\ln(e^3))^2}{2} - 3\ln(1) + \frac{(\ln(1))^2}{2} \quad \text{1pt.}$$

$$= 3 \cdot 3 - \frac{3^2}{2}$$

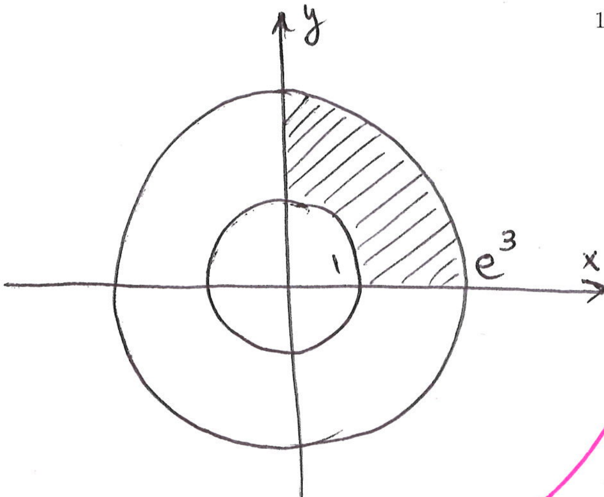
$$= \boxed{9/2} \quad \text{1pt.}$$

3. [20 points] Compute:

$$\iint_R \frac{\ln(x^2+y^2)}{x^2+y^2}\, dA,$$

where R is the region in the xy-plane given by:

$$1 \le x^2+y^2 \le e^6;\ x>0;\ y>0.$$



$$\iint_R \frac{\ln(x^2+y^2)}{x^2+y^2}\, dA$$

10 pts. $= \int_0^{\pi/2}\int_1^{e^3} r\cdot\frac{\ln(r^2)}{r^2}\, dr\, d\theta$

3pts. $= \int_0^{\pi/2}\int_1^{e^3} \frac{2\ln(r)}{r}\, dr\, d\theta$

3pts. $= \int_0^{\pi/2} \ln^2(r)\Big|_{r=1}^{r=e^3}\, d\theta$

2pts. $= \int_0^{\pi/2} (9)\, d\theta$

2pts. $= 9\theta\Big|_{\theta=0}^{\theta=\pi/2} = \boxed{\frac{9\pi}{2}}$

bounds for r 3pts.

bounds for θ 3pts.

$\frac{\ln(r^2)}{r^2}$ 2pts.

$\underline{r\,dr\,d\theta}$ 2pts.