Name: Solutions

October 29th, 2014.
Math 2401; Sections D1, D2, D3.
Georgia Institute of Technology
Exam 3

I commit to uphold the ideals of honor and integrity by refusing to betray the trust bestowed upon me as a member of the Georgia Tech community. By signing my name below I pledge that I have neither given nor received help on this exam.

Pledged: ______________________________

Problem	Possible Score	Earned Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

Remember that you must SHOW YOUR WORK to receive credit!

Good luck!

1. [20 points] Find the volume of the region in space bounded above by the surface $z = 4\cos(x)\sin(y)$ and below by the rectangle: $0 \le x \le \pi/6$, $0 \le y \le \pi/4$.

(2pts.) $V = \iint_R f(x,y)\,dA$

(6pts.) $= \int_0^{\pi/6}\int_0^{\pi/4} 4\cos(x)\sin(y)\,dy\,dx$ $\left(\text{or } \int_0^{\pi/4}\int_0^{\pi/6} 4\cos(x)\sin(y)\,dx\,dy\right)$

(2pts.) $= \int_0^{\pi/6} -4\cos(x)\cos(y)\Big|_{y=0}^{y=\pi/4} dx$ $\left(= \int_0^{\pi/4} 4\sin(x)\sin(y)\Big|_{x=0}^{x=\pi/6} dy\right)$

(2pts.) $= \int_0^{\pi/6}\left(-4\cos(x)\cdot\frac{\sqrt{2}}{2} + 4\cos(x)\right)dx$ $\left(= \int_0^{\pi/4} 4\cdot\frac{1}{2}\sin(y)\,dy\right)$

(2pts.) $= \int_0^{\pi/6}(4-2\sqrt{2})\cos(x)\,dx$ $\left(= \int_0^{\pi/4} 2\sin(y)\,dy\right)$

(2pts.) $= (4-2\sqrt{2})\sin(x)\Big|_{x=0}^{x=\pi/6}$ $\left(= -2\cos(y)\Big|_{y=0}^{y=\pi/4}\right)$

(2pts.) $= (4-2\sqrt{2})\cdot\frac{1}{2}$ $\left(= -2\cdot\frac{\sqrt{2}}{2} + 2\cdot 1\right)$

(2pts.) $= \boxed{2-\sqrt{2}}$ $\left(= \boxed{2-\sqrt{2}}\right)$

(2 pts.) Correct bounds for x

(2pts.) correct bounds for y

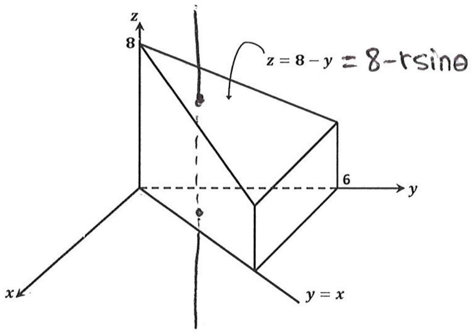
(2pts.) correct order of dx, dy.

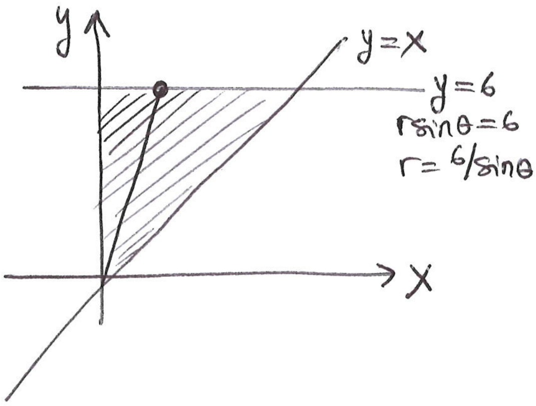
4. [20 points] a). Set up the integral for evaluating

[12pts.]

$$\int\int\int_D r\,dz\,dr\,d\theta$$

over the solid D shown in the figure below: D is the prism whose base is the triangle in the xy-plane bounded by the y-axis and the lines $y = x$, $y = 6$, and whose top lies in the plane $z = 8 - y$.





$$\boxed{\int_{\pi/4}^{\pi/2}\int_0^{\frac{6}{\sin\theta}}\int_0^{8-r\sin\theta} r\,dz\,dr\,d\theta}$$

4pts. - bounds for z

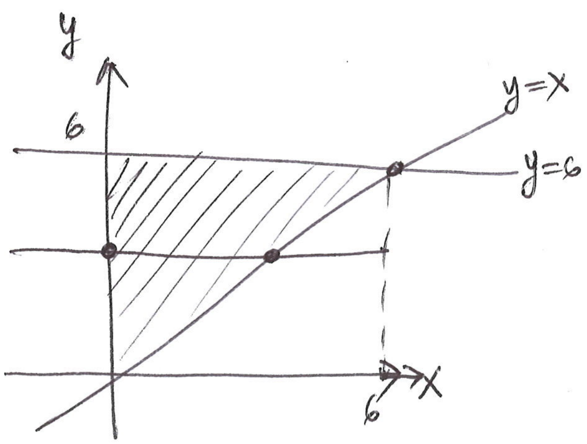
4pts. - bounds for r

4pts. - bounds for θ

[8pts.] b). Set up a double integral of the form:

$$\int\int_R f(x, y)\,dx\,dy$$

which would give the volume of the solid D in part a).



$$\boxed{\underbrace{\int_0^6}_{3pts.}\underbrace{\int_0^y}_{3pts.}\underbrace{(8-y)}_{2pts.}\,dx\,dy}$$

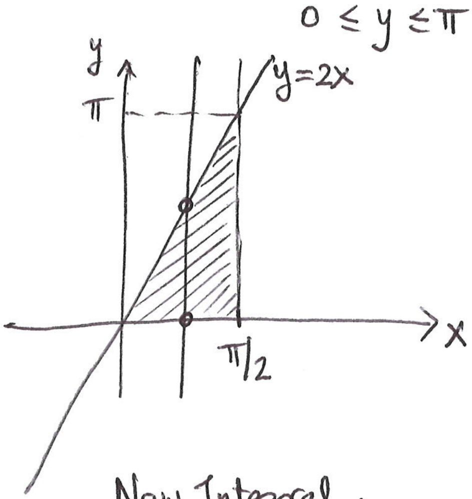
5. [20 points] Compute the integral:

$$\int_0^1 \int_0^{\pi} \int_{y/2}^{\pi/2} \frac{z\cos(x)}{x}\, dx\, dy\, dz.$$

2pts. Switch order of integration between x and y, since we cannot integrate $\frac{\cos(x)}{x}$ directly.

Region: $y/2 \le x \le \pi/2$

$0 \le y \le \pi$



Vertical Cross-Sections:

$$\int_0^{\pi/2}\int_0^{2x} dy\, dx$$

3pts.

3pts.

New Integral:

2pts.

2pts.

$$\int_0^1\int_0^{\pi/2}\int_0^{2x} \frac{z\cos(x)}{x}\, dy\, dx\, dz = \int_0^1\int_0^{\pi/2} \frac{z\cos(x)}{x}\, y\Big|_{y=0}^{y=2x} dx\, dz$$

2pts. $= \int_0^1\int_0^{\pi/2} \frac{z\cos(x)}{x}\cdot 2x\, dx\, dz = \int_0^1\int_0^{\pi/2} 2z\cos(x)\, dx\, dz$

2pts. $= \int_0^1 2z\sin(x)\Big|_{x=0}^{x=\pi/2} dz$

2pts. $= \int_0^1 2z(\sin\pi/2 - \sin 0)\, dz$

$= \int_0^1 2z\, dz$

2pts. $= z^2\Big|_0^1 = \boxed{1}$

Name: Solutions

November 19^{th}, 2014.
Math 2401; Sections D1, D2, D3.
Georgia Institute of Technology
Exam 4

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Pledged: ____________________

Problem	Possible Score	Earned Score
1	25	
2	25	
3	20	
4	15	
5	15	
Total	100	

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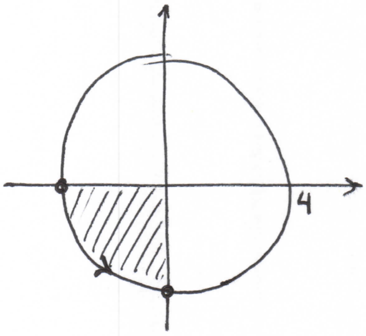
Good luck!

1. [25 points] Find:

$$\int_C f\, ds,$$

where $f(x, y) = x + y$, and C, positively oriented, is the quarter of the circle $x^2 + y^2 = 16$ lying in the third quadrant.



(4pts.) (3pts.)

$C: \vec{r}(t) = \langle 4\cos t, 4\sin t \rangle, \quad \pi \leq t \leq \frac{3\pi}{2}$

$x = 4\cos t$
$y = 4\sin t$

$f(\vec{r}(t)) = 4\cos t + 4\sin t$ (3pts.)

$\vec{v}(t) = \langle -4\sin t, 4\cos t \rangle$ (3pts.)

$|\vec{v}(t)| = \sqrt{16\sin^2 t + 16\cos^2 t} = 4$ (3pts.)

$$\int_C f\, ds = \int_{\pi}^{\frac{3\pi}{2}} f(\vec{r}(t))\, |\vec{v}(t)|\, dt$$ (2pts.)

$$= \int_{\pi}^{\frac{3\pi}{2}} (4\cos t + 4\sin t) \cdot 4\, dt$$ (2pts.)

$$= 16 (\sin t - \cos t) \Big|_{\pi}^{3\pi/2}$$ (3pts.)

$$= 16 [(-1-0) - (0-(-1))]$$

$$= \boxed{-32}$$ (2pts.)

2. [25 points] Recall Green's formulas:

$$\boxed{\oint_C \vec{F}\cdot\vec{n}\,ds = \oint_C M\,dy - N\,dx = \int\int_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)\,dA},$$

$$\boxed{\oint_C \vec{F}\cdot\vec{T}\,ds = \oint_C M\,dx + N\,dy = \int\int_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\,dA}.$$

Let the field:

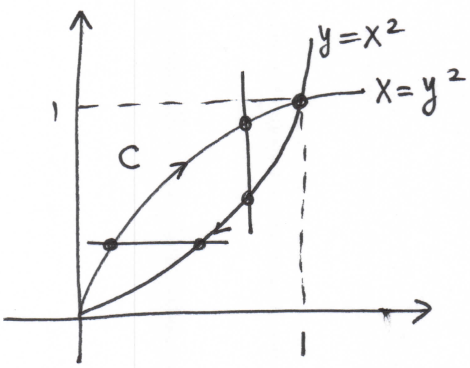
$$\vec{F} = (5xy + y^2)\vec{i} + (5x - y)\vec{j},$$

and let C be the boundary of the region in the first quadrant enclosed by the curves $x = y^2$ and $y = x^2$. Use Green's Theorem to fill in the blanks below, expressing the flux and circulation of $\vec{F}$ over C as double integrals over the region enclosed by C in the plane.

Flux of $\vec{F}$ over $C = \int_{\boxed{0}}^{\boxed{1}} \int_{\boxed{x^2}}^{\boxed{\sqrt{x}}} \boxed{5y-1}\,dy\,dx.$ (12 pts.) (8pts.) bounds (2 pts. each) (4pts.) $5y-1$

Circulation of $\vec{F}$ over $C = \int_{\boxed{0}}^{\boxed{1}} \int_{\boxed{y^2}}^{\boxed{\sqrt{y}}} \boxed{5-5x-2y}\,dx\,dy.$ (12pts.) (8pts.) bounds (2 pts. each) (4pts.) $5-5x-2y$

Be careful at the order of integration!
Do not compute the values of the integrals.



$M = 5xy + y^2$
$N = 5x - y$ (1pt.)

$$\text{Flux} = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA$$
$$= \iint_R (5y-1)\,dA$$
$$= \int_0^1 \int_{x^2}^{\sqrt{x}} (5y-1)\,dy\,dx$$

$$\text{Circulation} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$
$$= \iint_R \left(5 - (5x+2y)\right) dA$$
$$= \int_0^1 \int_{y^2}^{\sqrt{y}} (5-5x-2y)\,dx\,dy$$

3. [20 points] The following field:

$$\vec{F}(x,y,z) = \left\langle 2xye^{x^2y},\ \ x^2e^{x^2y} + \ln(z),\ \ \frac{y}{z} \right\rangle$$

is conservative.

(10 pts.) a). Find a potential function f for this field.

$$\frac{\partial f}{\partial x} = 2xye^{x^2y} \Rightarrow f = e^{x^2y} + g(y,z) \quad \text{(3pts.)}$$

$$\left.\begin{aligned} \Rightarrow \frac{\partial f}{\partial y} &= x^2e^{x^2y} + \frac{\partial g}{\partial y} \\ &= x^2e^{x^2y} + \ln(z) \end{aligned}\right\} \Rightarrow \frac{\partial g}{\partial y} = \ln(z) \Rightarrow g = y\ln(z) + h(z)$$

(3pts.)

$$f = e^{x^2y} + y\ln(z) + h(z)$$

$$\left.\begin{aligned} \Rightarrow \frac{\partial f}{\partial z} &= \frac{y}{z} + h'(z) \\ &= \frac{y}{z} \end{aligned}\right\} \Rightarrow h'(z) = 0 \Rightarrow h(z) = C \quad \text{(2pts.)}$$

$$\boxed{f(x,y,z) = e^{x^2y} + y\ln(z) + C} \quad \text{(2pts.)}$$

(5pts.) b). Compute $\int_C \vec{F} \cdot d\vec{r}$, where C is a smooth path from $(1,0,3)$ to $(0,1,e^3)$.

$$\int_C \vec{F}\cdot d\vec{r} = f(0,1,e^3) - f(1,0,3) \quad \text{(3pts.)}$$

$$= (1+3) - (1+0)$$

$$= \boxed{3} \quad \text{(2pts.)}$$

(5pts.) c). Compute $\oint_C \vec{F} \cdot d\vec{r}$, where C is a simple closed curve (loop).

$\oint_C \vec{F}\cdot d\vec{r} = \boxed{0}$ because $\vec{F}$ is conservative. (5pts.)

4. [15 points] Find the area of the region enclosed by the sinusoidal curve:

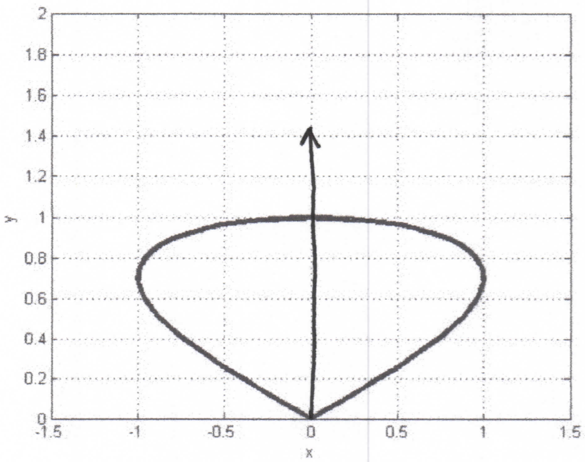
$$\vec{r}(t) = \sin(2t)\vec{i} + \sin(t)\vec{j},\ 0 \le t \le \pi$$

in the xy-plane, pictured below. You may use Green's area formula:

$$A = \frac{1}{2}\oint_C x\,dy - y\,dx,$$

or any other method you like. You may need to recall that:

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta) \text{ and } \cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = 2\cos^2(\theta) - 1 = 1 - 2\sin^2(\theta).$$



Method 1: Green's Theorem

$x = \sin(2t)$; (1pt.)

$y = \sin(t)$; (1pt.)

$dx = 2\cos(2t)$; (2pts.)

$dy = \cos(t)$ (2pts.)

$$A = \frac{1}{2}\int_0^{\pi}\left[\sin(2t)\cos(t) - \sin(t)\cdot 2\cos(2t)\right]dt \quad \text{(2pts.)}$$

$$= \frac{1}{2}\int_0^{\pi}\left[2\sin(t)\cos^2(t) - 2\sin(t)(2\cos^2 t - 1)\right]dt$$

$$= \int_0^{\pi}\left[\sin(t)\cos^2(t) - 2\sin(t)\cos^2(t) + \sin(t)\right]dt$$

$$= \int_0^{\pi}\left[-\sin(t)\cos^2(t) + \sin(t)\right]dt$$

$$= \left(\frac{\cos^3(t)}{3} - \cos(t)\right)\Bigg|_0^{\pi}$$

$$= \left(\frac{-1}{3} - (-1)\right) - \left(\frac{1}{3} - 1\right) = \frac{2}{3} + \frac{2}{3}$$

(6pts.) computation

$$= \boxed{4/3} \quad \text{(1pt.)}$$

Method 2 : $\text{Area} = \iint_R dA$

$x = \sin(2t)$

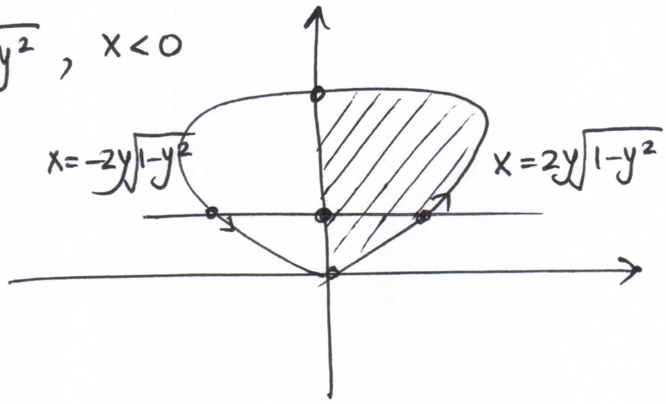
$y = \sin(t), \quad 0 \leq t \leq \pi \quad \Rightarrow \boxed{y \geq 0}$

$$x = 2\sin(t)\cos(t) \Rightarrow x = \begin{cases} 2y\sqrt{1-y^2}, & x \geq 0 \\ -2y\sqrt{1-y^2}, & x < 0 \end{cases}$$

$= 2y\cos(t)$

$= 2y\left(\pm\sqrt{1-\sin^2(t)}\right)$

(9 pts.) $= 2y\left(\pm\sqrt{1-y^2}\right)$



(2pts.) $A = 2\int_0^1 \int_0^{2y\sqrt{1-y^2}} dx\,dy$

$= 2\int_0^1 2y\sqrt{1-y^2}\,dy$

$= 2 \cdot \frac{-2}{3}(1-y^2)^{3/2}\Big|_0^1 = -\frac{4}{3}(0-1)$

Computation (3pts.)

$= \boxed{4/3}$ (1pt.)