Worksheet 1 - Review

1. Find all the values of x where the tangent lines to $y = \frac{x^3}{3} + 5x$ and $y = 3x^2$ are parallel.

Recall: - Slope of tangent line to a graph is given by the derivative
- Two lines are parallel when their slopes are equal

$y = \frac{x^3}{3} + 5x \Rightarrow y' = x^2 + 5$

$y = 3x^2 \Rightarrow y' = 6x$

$x^2 + 5 = 6x$

$x^2 - 6x + 5 = 0$

$(x-5)(x-1) = 0 \Rightarrow \boxed{x = 1, 5}$

2. Given that $f'(x) = 2e^x - 3$, and that $f(0) = 7$, find $f(x)$.

$f(x) = \int (2e^x - 3) dx = 2e^x - 3x + C$

$\left. \begin{array}{l} f(0) = 2e^0 - 3 \cdot 0 + C = 2 + C \\ f(0) = 7 \end{array} \right\} \Rightarrow C = 5 \Rightarrow \boxed{f(x) = 2e^x - 3x + 5}$

3. Find $\frac{dy}{dx}$ for $y = x^{3x}$.

$y = x^{3x} = \left(e^{\ln(x)} \right)^{3x} = e^{\ln(x) \cdot 3x}$

$y' = e^{\ln(x) \cdot 3x} \left[\ln(x) \cdot 3x \right]' = x^{3x} \left[\frac{1}{x} \cdot 3x + \ln(x) \cdot 3 \right]$

$= \boxed{x^{3x} \left[3 + 3\ln(x) \right]}$

4. For what value of c is the function below continuous on $(-\infty, \infty)$?

$$f(x) = \begin{cases} cx + 4 & \text{, if } x < 1 \\ x^2 + 2c & \text{, if } x \geq 1 \end{cases}$$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (cx+4) = c+4$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x^2+2c) = 1+2c = f(1)$$

$$c+4 = 1+2c \quad \Rightarrow \boxed{c=3}$$

Recall: A function is continuous at a point a in the domain if and only if

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$$

5. Find the derivative of $f(x) = \arctan(\sin(x))$.

$$f'(x) = \frac{\cos(x)}{1+\sin^2(x)}$$

Chain Rule

6. Find the derivative of $f(x) = 3^{x^2 - 3x + 4}$.

$$f'(x) = 3^{x^2-3x+4} \cdot \ln(3) \cdot (2x-3)$$

Chain Rule

7. Find the integral:

$$\int_1^{e^4} \frac{1}{2x\sqrt{\ln(x)}} dx = \int_0^4 \frac{1}{2\sqrt{u}} du = \sqrt{u} \Big|_0^4 = \boxed{2}$$

$u = \ln(x)$

$du = \frac{1}{x} dx$

Substitution

$x = 1 \Rightarrow u = \ln(1) = 0$

$x = e^4 \Rightarrow u = \ln(e^4) = 4$

8. Find the integral:

$$\int \frac{x-2}{x^2 - 6x + 10} dx = \int \frac{x-2}{(x-3)^2 + 1} dx = \int \frac{(x-3)+1}{(x-3)^2+1} dx$$

$$= \int \frac{x-3}{(x-3)^2+1} dx + \int \frac{1}{(x-3)^2+1} dx = \boxed{\frac{1}{2}\ln(x^2-6x+10) + \arctan(x-3) + C}$$

$u = (x-3)^2 + 1$

$du = 2(x-3)dx$

$\frac{1}{2}du = (x-3)dx$

$$\Rightarrow \int \frac{1}{u} \cdot \frac{1}{2} du$$

$$= \frac{1}{2}\ln|u| = \frac{1}{2}\ln\left|(x-3)^2+1\right| + C$$

$$= \frac{1}{2}\ln(x^2-6x+10) + C$$

9. Find the integral:

$$\int x^6 \ln(x)\,dx = \int \left(\frac{x^7}{7}\right)' \ln(x)\,dx = \frac{x^7}{7}\ln(x) - \int \frac{x^7}{7}(\ln(x))'\,dx$$

$$= \frac{x^7}{7}\ln(x) - \int \frac{x^7}{7}\cdot\frac{1}{x}\,dx = \frac{x^7}{7}\ln(x) - \frac{1}{7}\int x^6\,dx$$

$$= \frac{x^7}{7}\ln(x) - \frac{1}{7}\cdot\frac{1}{7}x^7 + C$$

$$= \boxed{\frac{x^7}{7}\ln(x) - \frac{x^7}{49} + C} \qquad \underline{\text{Integration by Parts}}$$

10. Find the integral:

$$\int \pi e^{25}\,dx = \boxed{\pi e^{25} x + C}$$

11. Find the integral:

$$\int \frac{\sin^3(x)}{\cos(x)}\,dx = \int \frac{\sin^2(x)}{\cos(x)}\cdot\sin(x)\,dx = \int \frac{1-\cos^2(x)}{\cos(x)}\cdot\sin(x)\,dx$$

$$= \int \frac{1-u^2}{u}(-du) \qquad\qquad u = \cos(x)$$
$$\qquad\qquad\qquad\qquad\qquad\qquad du = -\sin(x)\,dx$$

$$= -\int\left(\frac{1}{u} - u\right)du$$

$$= -\left(\ln|u| - u^2/2\right) + C$$

$$= \boxed{-\ln|\cos(x)| + \frac{\cos^2(x)}{2} + C}$$

12. Find the integral:

$$\int \arctan(x)\,dx = \int (x)'\arctan(x)\,dx = x\arctan(x) - \int x(\arctan'(x))\,dx$$

$$= x\arctan(x) - \int \frac{x}{1+x^2}\,dx$$

$$= x\arctan(x) - \int \frac{1}{u}\cdot\frac{1}{2}\,du \qquad \boxed{\begin{array}{l} u = 1+x^2 \\ du = 2x\,dx \\ \frac{1}{2}du = x\,dx \end{array}}$$

$$= x\arctan(x) - \frac{1}{2}\ln|u| + C$$

$$= \boxed{x\arctan(x) - \frac{1}{2}\ln(1+x^2) + C}$$

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13. Find the integral:

$$\int \frac{1}{2\sqrt{x+1} + x\sqrt{x+1}}dx = \int \frac{1}{\sqrt{x+1}\,(x+2)}dx = \int \frac{1}{\sqrt{x+1}\left(1 + (\sqrt{x+1})^2\right)}dx$$

$$u = \sqrt{x+1}$$
$$du = \frac{1}{2\sqrt{x+1}}dx$$
$$2du = \frac{1}{\sqrt{x+1}}dx$$

$$= \int \frac{1}{1+u^2}\, 2du$$

$$= 2\arctan(u) + C$$

$$= \boxed{2\arctan(\sqrt{x+1}) + C}$$

14. Find the integral:

$$\int \cos(\sqrt{x})dx$$

$$u = \sqrt{x}$$
$$u^2 = x$$
$$2u\,du = dx$$

$$\int \cos(\sqrt{x})dx = \int \cos(u)\cdot 2u\,du = 2\int u\cos(u)du$$

$$= 2\int u\big(\sin(u)\big)'du$$

$$= 2\left[u\sin(u) - \int u'\sin(u)du\right]$$

$$= 2\left[u\sin(u) - \int \sin(u)du\right]$$

$$= 2\left[u\sin(u) + \cos(u) + C\right]$$

$$= \boxed{2\sqrt{x}\sin(\sqrt{x}) + 2\cos(\sqrt{x}) + C}$$