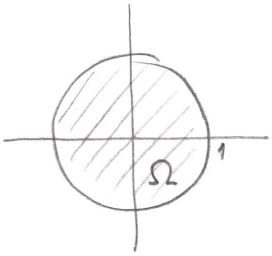
① $f(x,y)=2x^2+x+2y^2-2$; $\Omega=\{(x,y): x^2+y^2\le 1\}$



<u>Critical Points:</u> $f_x = 4x+1 \quad x=-1/4$

$f_y = 4y \quad y=0$

$(-1/4, 0)$

$$f(-1/4,0)=\frac{2}{16}-\frac{1}{4}-2$$

<u>Boundary</u> $\boxed{x^2+y^2=1}$

<u>Solution I</u>: Parametrize the boundary

$$\begin{cases} x=\cos\theta \\ y=\sin\theta \end{cases} \quad 0\le\theta\le 2\pi$$

$$f(\theta)=2\cos^2\theta+\cos\theta+2\sin^2\theta-2=\cos\theta$$

$$f'(\theta)=-\sin\theta$$

$$f'(\theta)=0 \Rightarrow \theta\in\{0,\pi,2\pi\}$$

$f(0)=f(2\pi)=1 \qquad f(\pi)=-1$

$\theta\in\{0,2\pi\}: (x,y)=(1,0) \qquad \theta=\pi \Rightarrow (x,y)=(-1,0)$

(x,y)	$f(x,y)$	
$(-1/4,0)$	$-17/8$	← min
$(1,0)$	1	← max
$(-1,0)$	-1	

Absolute min: $-17/8$ at $(-1/4,0)$

Absolute max: 1 at $(1,0)$.

② $f(x,y)=(x^2+y^2)e^{-x}$

<u>Critical Points</u>: $f_x = 2xe^{-x}-(x^2+y^2)e^{-x}=(2x-x^2-y^2)e^{-x}$

$f_y = 2ye^{-x}$

$$\begin{cases} 2x-x^2-y^2=0 \Rightarrow 2x-x^2=0 \Rightarrow x(2-x)=0 \Rightarrow x\in\{0,2\} \\ 2y=0 \Rightarrow y=0 \end{cases}$$

$(0,0)$ & $(2,0)$

$$f_{xx}=(2-2x)e^{-x}-(2x-x^2-y^2)e^{-x}=(2-4x+x^2+y^2)e^{-x}$$

$$f_{yy}=2e^{-x}$$

$$f_{xy}=-2ye^{-x}$$

$$\Delta f=2(2-4x+x^2+y^2)e^{-2x}-4y^2e^{-2x}=2(2-4x+x^2-y^2)e^{-2x}$$

$\Delta f(0,0)=4>0$

$f_{xx}(0,0)=2>0 \Rightarrow \boxed{(0,0) \text{ is a local min}}$

$\Delta f(2,0)=2(2-8+4)e^{-4}<0 \Rightarrow \boxed{(2,0) \text{ is a saddle point}}$

③ $f(x,y) = y^2x - yx^2 + xy$

$f_x = y^2 - 2xy + y$

$f_y = 2xy - x^2 + x$

$\Rightarrow$ Solve $\begin{cases} y(y-2x+1)=0 & (1) \\ x(2y-x+1)=0 & (2) \end{cases}$

(1) $\Rightarrow$ either $y=0$ or $y-2x+1=0$

Case 1: $y=0$ $\Rightarrow$ (2) becomes $x(-x+1)=0 \Rightarrow x=0$ or $x=1$ $\Rightarrow$ $(0,0)$ & $(1,0)$

Case 2: $y-2x+1=0$ $\Rightarrow$ System becomes $\begin{cases} y-2x+1=0 & (1') \\ x(2y-x+1)=0 & (2) \end{cases}$

From (2): either $x=0$ or $2y-x+1=0$

Case 2a: $x=0$ $\Rightarrow$ (1') becomes: $y+1=0 \Rightarrow y=-1$ $\qquad (0,-1)$

Case 2b: $2y-x+1=0$ $\Rightarrow \begin{cases} y-2x+1=0 \\ 2y-x+1=0 \end{cases} \Rightarrow \left.\begin{matrix} 2y=x-1 \\ 2y=4x-2 \end{matrix}\right\} \Rightarrow$

$x-1=4x-2 \Rightarrow 1=3x \Rightarrow x=1/3$

$\Rightarrow y=-1/3$

$(1/3,-1/3)$

C.Pts: $\boxed{(0,0), (1,0), (0,-1), (1/3,-1/3)}$

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④ $f(x,y) = \ln x + 2\ln y - x - 4y$

$f_x = \frac{1}{x} - 1$

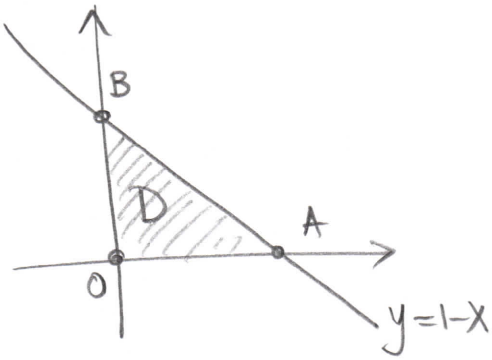
$f_y = \frac{2}{y} - 4$

$\Rightarrow \begin{cases} \frac{1-x}{x} = 0 & \Rightarrow x=1 \\ \frac{1-2y}{y} = 0 & \Rightarrow y=1/2 \end{cases}$

C.Pt.: $(1, 1/2)$

$\left.\begin{matrix} f_{xx} = -\frac{1}{x^2} \\ f_{yy} = -\frac{2}{y^2} \\ f_{xy} = 0 \end{matrix}\right\} \Rightarrow \Delta_f = \frac{2}{x^2y^2} \Rightarrow \begin{matrix} \Delta_f(1,1/2) > 0 \\ f_{xx}(1,1/2) < 0 \end{matrix} \Rightarrow \boxed{(1,1/2) = \text{local max}}$

⑤ $f(x,y) = x^3 + x^2 y + 2y^2$; $D = \{(x,y) : x, y \geq 0 ; x+y \leq 1\}$



$f_x = 3x^2 + 2xy = x(3x+2y)$

$f_y = x^2 + 4y$

$$\begin{cases} x(3x+2y) = 0 \Rightarrow \text{either } x=0 \text{ or } 3x+2y=0 \\ x^2+4y=0 \end{cases}$$

$x=0 \Rightarrow 4y=0 \Rightarrow y=0$ → $\cancel{(0,0)}$ not in the interior of D (but will evaluate here later anyway)

$3x+2y=0 \Rightarrow 2y=-3x \Rightarrow 4y=-6x$

$\Rightarrow x^2-6x=0 \Rightarrow x(x-6)=0$

$x=0 \Rightarrow y=0$: $\cancel{(0,0)}$

$x=6 \Rightarrow y=-9$: $\cancel{(6,-9)}$ → not in the interior of D!

$\Rightarrow$ No critical points in the interior of D.

Boundary:

$\boxed{OA}$ $y=0$; $x \in [0,1]$

$f(x) = x^3 \Rightarrow f'(x) = 3x^2$

$f'(x)=0 \Rightarrow x=0 \Rightarrow (0,0)$

$f(1,0)=1$

$\boxed{OB}$ $x=0$, $y \in [0,1]$

$f(y) = 2y^2$

$f'(y) = 4y$; $f'(y)=0 \Rightarrow y=0 \Rightarrow (0,0)$

$f(0,1)=2$

$\boxed{AB}$ $y=1-x$; $x \in [0,1]$

$f(x) = x^3 + x^2(1-x) + 2(1-x)^2$

$= \cancel{x^3} + x^2 - \cancel{x^3} + 2x^2 - 4x + 2$

$= 3x^2 - 4x + 2$

$f'(x) = 6x - 4$

$f'(x) = 0 \Rightarrow x = 2/3 \Rightarrow y = 1/3$

$f(2/3, 1/3) = 18/27$

(x,y)	$f(x,y)$
$(0,0)$	0 ← min
$(0,1)$	2 ← max
$(1,0)$	1
$(2/3,1/3)$	18/27

Max = 2 at $(0,1)$

Min = 0 at $(0,0)$.