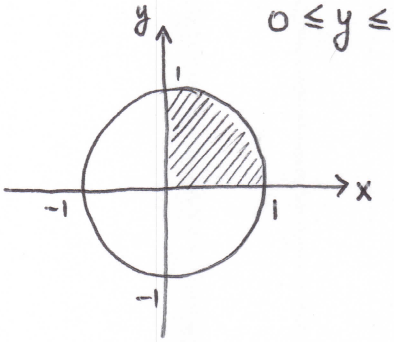
① $\int_0^1 \int_0^{\sqrt{1-y^2}} \cos(x^2+y^2)\,dx\,dy$

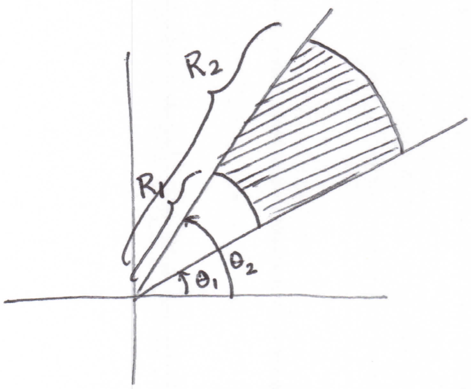
$0 \le x \le \sqrt{1-y^2}$
$0 \le y \le 1$



Polar Coordinates:

$$\int_0^{\pi/2}\int_0^1 \cos(r^2)\cdot r\,dr\,d\theta$$
$$= \int_0^{\pi/2}\left(\frac{1}{2}\sin(r^2)\Big|_{r=0}^{r=1}\right)d\theta$$
$$= \int_0^{\pi/2} \frac{1}{2}\sin(1)\,d\theta$$
$$= \frac{1}{2}\sin(1)\theta\Big|_{\theta=0}^{\theta=\pi/2} = \boxed{\frac{\pi}{4}\sin(1)}$$

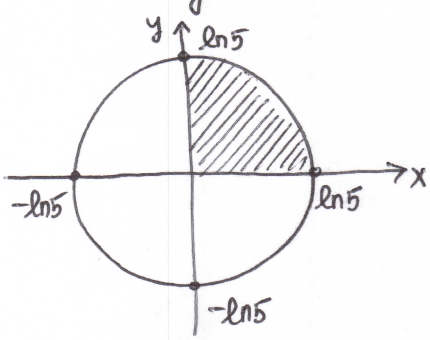
② Area of a circular section $R_1 \le r \le R_2$; $\theta_1 \le \theta \le \theta_2$:



$$A = \int_{\theta_1}^{\theta_2}\int_{R_1}^{R_2} r\,dr\,d\theta$$
$$= \int_{\theta_1}^{\theta_2} \frac{r^2}{2}\Big|_{r=R_1}^{R_2} d\theta$$
$$= \int_{\theta_1}^{\theta_2} \frac{R_2^2-R_1^2}{2}\,d\theta$$
$$= \frac{R_2^2-R_1^2}{2}\theta\Big|_{\theta=\theta_1}^{\theta_2} = \boxed{\frac{1}{2}(R_2^2-R_1^2)(\theta_2-\theta_1)}$$

③ $\int_0^{\ln 5}\int_0^{\sqrt{(\ln 5)^2-y^2}} e^{\sqrt{x^2+y^2}}\,dx\,dy$

$0 \le x \le \sqrt{(\ln 5)^2-y^2}$
$0 \le y \le \ln 5$



Polar Coordinates:

$$\int_0^{\pi/2}\int_0^{\ln 5} e^r\cdot r\,dr\,d\theta$$
$$= \int_0^{\pi/2}\int_0^{\ln 5} (e^r)'\cdot r\,dr\,d\theta$$
$$= \int_0^{\pi/2}\left(e^r\cdot r\Big|_0^{\ln 5} - \int_0^{\ln 5} e^r\,dr\right)d\theta$$
$$= \int_0^{\pi/2}\left(5\ln 5 - e^r\Big|_0^{\ln 5}\right)d\theta$$
$$= \int_0^{\pi/2}(5\ln 5 - 5 + 1)\,d\theta = \boxed{\frac{\pi}{2}(5\ln 5 - 4)}$$

④ Area of region enclosed by the curves:

$$r=\cos\theta \quad \text{and} \quad r=\sin\theta$$

What do these curves look like?

$$r=\cos\theta \quad | \cdot r$$
$$r^2 = r\cos\theta$$
$$x^2+y^2 = x$$
$$x^2 - x + y^2 = 0$$

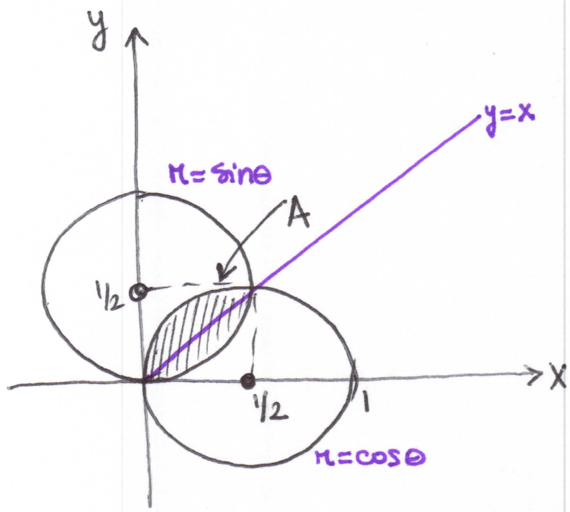
Complete the Square:

$$x^2 - 2\cdot\frac{1}{2}x + y^2 = 0$$
$$x^2 - 2\cdot\frac{1}{2}x + \frac{1}{4} + y^2 = \frac{1}{4}$$
$$\left(x-\frac{1}{2}\right)^2 + y^2 = \left(\frac{1}{2}\right)^2$$

Circle centered at $\left(\frac{1}{2},0\right)$, w/ radius $\frac{1}{2}$.

$$r=\sin\theta \quad | \cdot r$$
$$r^2 = r\sin\theta$$
$$x^2+y^2 = y$$
$$x^2+y^2-y=0$$
$$x^2+y^2-2\cdot\frac{1}{2}y+\frac{1}{4}=\frac{1}{4}$$
$$x^2+\left(y-\frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2$$

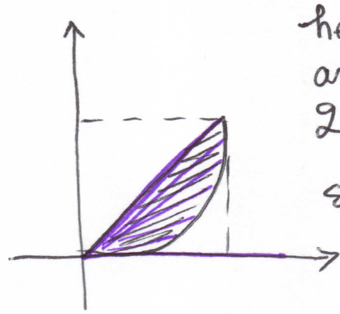
Circle centered at $(0, 1/2)$ w/ radius $1/2$.



Thus the area of the original region is

$$\boxed{\frac{\pi-2}{8}}$$

To find the area of the region, we look at a half of the region:



here θ varies from 0 to $\pi/4$ and r "ends" on the quarter of the circle $r=\sin\theta$, so the area of this half-region is:

$$\int_0^{\pi/4}\int_0^{\sin\theta} r\,dr\,d\theta = \int_0^{\pi/4} \left.\frac{r^2}{2}\right|_{r=0}^{\sin\theta} d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} \sin^2\theta\, d\theta$$

$$\boxed{\begin{aligned} \sin^2\theta &= \frac{1-\cos(2\theta)}{2} \\ \cos^2\theta &= \frac{1+\cos(2\theta)}{2} \end{aligned}}$$

$$= \frac{1}{2}\int_0^{\pi/4} \frac{1-\cos(2\theta)}{2}\, d\theta$$

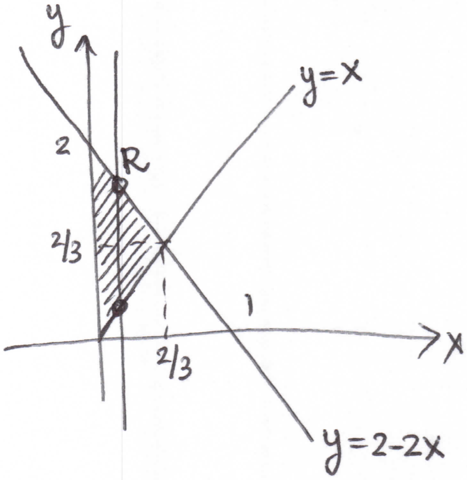
$$= \frac{1}{4}\left(\theta - \frac{1}{2}\sin(2\theta)\right)\Bigg|_{\theta=0}^{\pi/4}$$

$$= \frac{1}{4}\left(\frac{\pi}{4} - \frac{1}{2}\sin(\pi/2)\right) = \frac{1}{4}\left(\frac{\pi}{4}-\frac{1}{2}\right) = \frac{\pi-2}{16}$$

⑤ $$\int_{\pi/4}^{\pi/2}\int_0^{\frac{2}{\sin\theta+2\cos\theta}} r^3\cos^2\theta \, dr\, d\theta$$

Sketch the region:

$$0 \le r \le \frac{2}{\sin\theta + 2\cos\theta}$$



$$r = \frac{2}{\sin\theta + 2\cos\theta}$$

$$r\sin\theta + 2r\cos\theta = 2$$

$$y + 2x = 2$$

$$\boxed{y = 2 - 2x}$$

Point of intersection:

$$x = 2 - 2x$$

$$3x = 2$$

$$x = 2/3, \; y = 2/3$$

Convert integral to rectangular:

$$\int_{\pi/4}^{\pi/2}\int_0^{\frac{2}{\sin\theta+2\cos\theta}} \underbrace{r^3\cos^2\theta \, dr\, d\theta}_{\underbrace{r^2\cos^2\theta}_{x^2}\ \underbrace{(r\,dr\,d\theta)}_{dy\,dx}} = \int_0^{2/3}\int_x^{2-2x} x^2 \, dy\, dx$$

$$= \int_0^{2/3} x^2 y \Big|_{y=x}^{y=2-2x} dx = \int_0^{2/3} \left(x^2(2-2x) - x^3\right) dx$$

$$= \int_0^{2/3} \left(2x^2 - 3x^3\right) dx = \left(\frac{2}{3}x^3 - \frac{3}{4}x^4\right)\Big|_0^{2/3} = \left(\frac{2}{3}\right)^4 - \frac{3}{4}\left(\frac{2}{3}\right)^4 = \frac{1}{4}\left(\frac{2}{3}\right)^4$$

$$= \boxed{\frac{4}{81}}$$