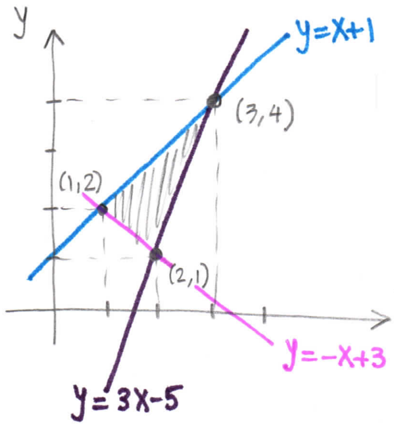
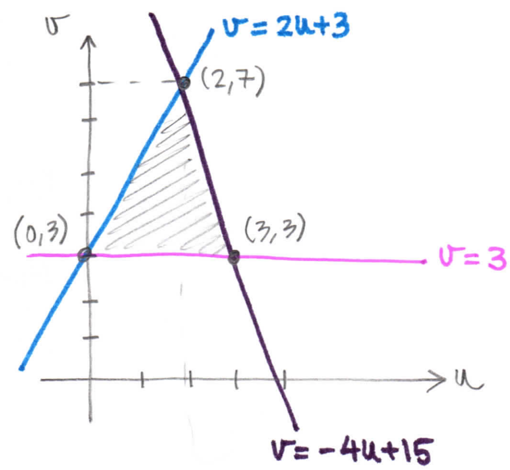
1. Triangle with vertices $(1,2), (2,1), (3,4)$ in x,y-plane

Transformation: $\begin{cases} u = 2x - y \\ v = x + y \end{cases} \Rightarrow \begin{array}{l} 3x = u + v \\ 3y = 2v - u \end{array} \Rightarrow \begin{cases} x = \frac{1}{3}(u+v) \\ y = \frac{1}{3}(-u+2v) \end{cases}$





Line through $(1,2)$ and $(2,1)$: $y = -x + 3$

Under transformation: $\frac{1}{3}(-u+2v) = -\frac{1}{3}(u+v) + 3$

$\cancel{-u} + 2v = \cancel{-u} - v + 9 \Rightarrow 3v = 9 \Rightarrow \boxed{v = 3}$

Line through $(1,2)$ and $(3,4)$: $y = x + 1$

Transform: $\frac{1}{3}(-u+2v) = \frac{1}{3}(u+v) + 1$

$-u + 2v = u + v + 3 \Rightarrow \boxed{v = 2u + 3}$

Line through $(2,1)$ and $(3,4)$: $y = 3x - 5$

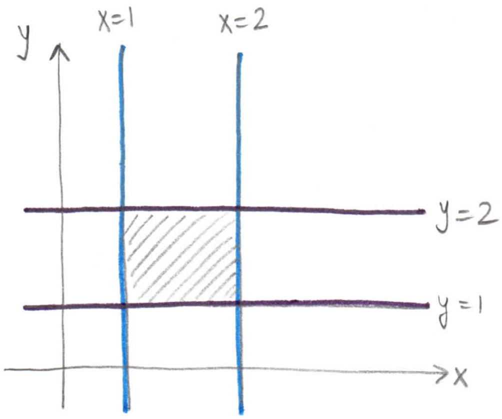
Transform: $\frac{1}{3}(-u+2v) = u + v - 5$

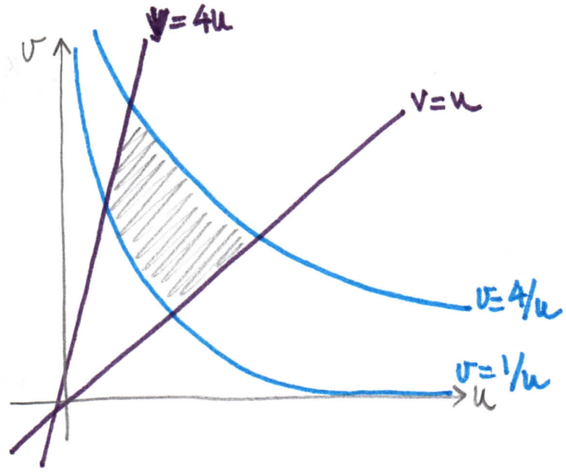
$-u + 2v = 3u + 3v - 15$

$\boxed{v = -4u + 15}$

<u>Remark</u>: This is just the triangle with vertices $(0,3), (3,3), (2,7)$, which are the images of $(1,2), (2,1), (3,4)$ under the transformation! This is typical for <u>linear transformations</u> (transform lines into lines).

② Image of $[1,2]\times[1,2]$ under $\begin{cases} u = \dfrac{x}{y} \\ v = xy \end{cases}$; $x, y > 0$.



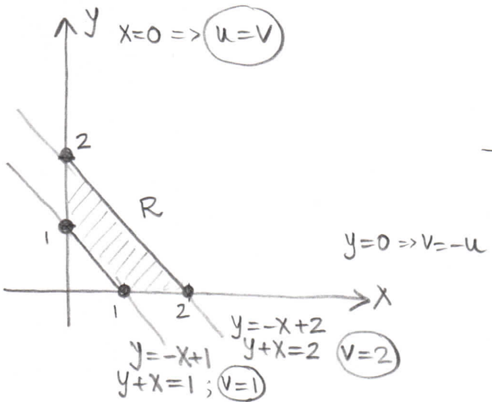


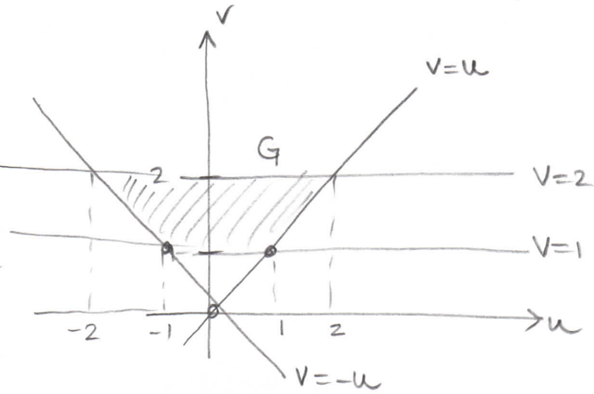
$x = uy \Rightarrow v = uy^2 \Rightarrow y = \sqrt{v/u} \Rightarrow x = \sqrt{uv}$

$\begin{cases} x = \sqrt{uv} \\ y = \sqrt{\frac{v}{u}} \end{cases}$

Boundary curve of R in x,y-plane:	Transformed curve in u,v-plane:
$x=1$	$\sqrt{uv}=1 \Rightarrow uv=1 \Rightarrow v=1/u$
$x=2$	$\sqrt{uv}=2 \Rightarrow uv=4 \Rightarrow v=4/u$
$y=1$	$\sqrt{\frac{v}{u}}=1 \Rightarrow v=u$
$y=2$	$\sqrt{\frac{v}{u}}=2 \Rightarrow v=4u$

3 $\iint_R \cos\left(\frac{y-x}{y+x}\right) dA$; R: trapezoidal region with vertices $(1,0), (2,0), (0,2), (0,1)$.





Substitution: $\begin{cases} u = y - x \\ v = y + x \end{cases} \Rightarrow \begin{cases} x = \frac{1}{2}(-u+v) \\ y = \frac{1}{2}(u+v) \end{cases} \Rightarrow J = \begin{vmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{vmatrix} = -\frac{1}{2}$

$2y = u+v$

$2x = -u+v$

$$\iint_R \cos\left(\frac{y-x}{y+x}\right) dA = \iint_G \cos\left(\frac{u}{v}\right)\cdot\frac{1}{2}\, dA = \frac{1}{2}\int_1^2\int_{-v}^{v} \cos\left(\frac{u}{v}\right) du\, dv$$

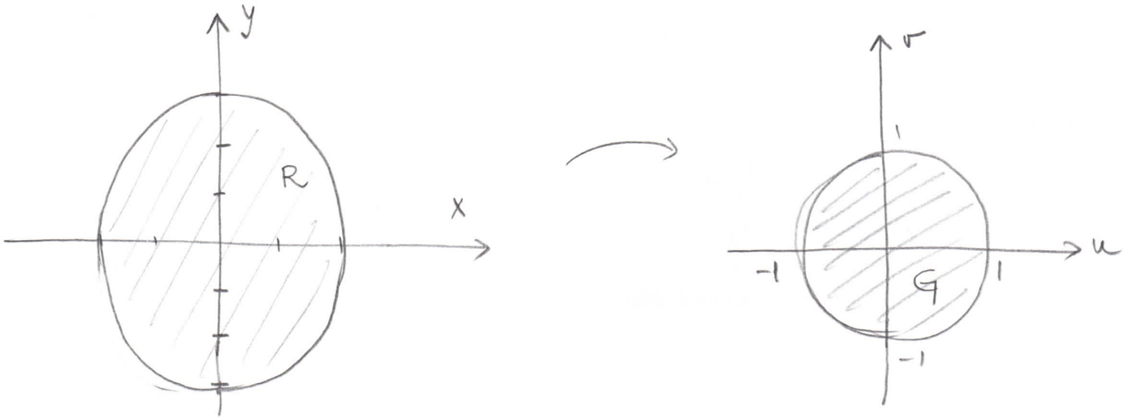
$$= \frac{1}{2}\int_1^2 v\sin\left(\frac{u}{v}\right)\Big|_{u=-v}^{v} dv$$

$$= \frac{1}{2}\int_1^2 \left(v\sin(1) - v\sin(-1)\right) dv$$

$$= \int_1^2 v\sin(1)\, dv = \frac{v^2}{2}\Big|_1^2 \sin(1) = \boxed{\frac{3}{2}\sin(1)}$$

④. $\iint_R x^2\,dA$; R = interior of the ellipse $9x^2+4y^2=36$

$$\frac{x^2}{4}+\frac{y^2}{9}=1 \qquad \left(\frac{x}{2}\right)^2+\left(\frac{y}{3}\right)^2=1$$



<u>Substitution</u> : $\begin{cases} u = x/2 \\ v = y/3 \end{cases} \Rightarrow \begin{cases} x = 2u \\ y = 3v \end{cases} \Rightarrow J = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6$

Under the transformation: $\left(\frac{x}{2}\right)^2+\left(y/3\right)^2=1$ becomes $u^2+v^2=1$ - unit circle!

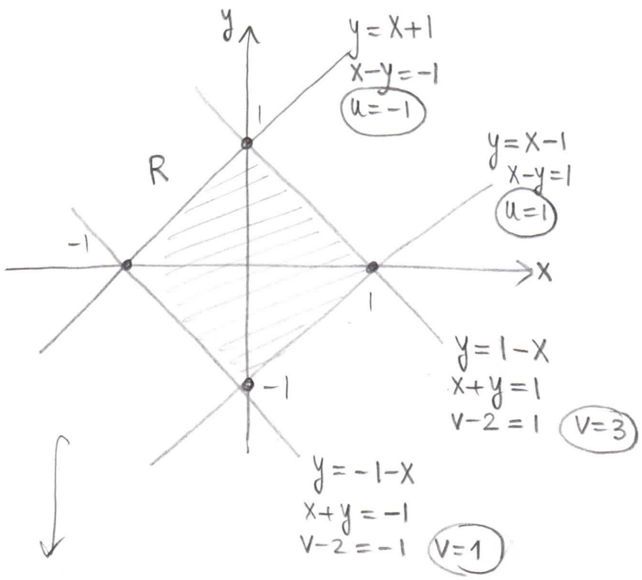
$$\Rightarrow \iint_R x^2\,dA = \iint_G 4u^2\cdot 6\,dA = 24\int_0^{2\pi}\int_0^1 r^2\cos^2\theta\; r\,dr\,d\theta$$

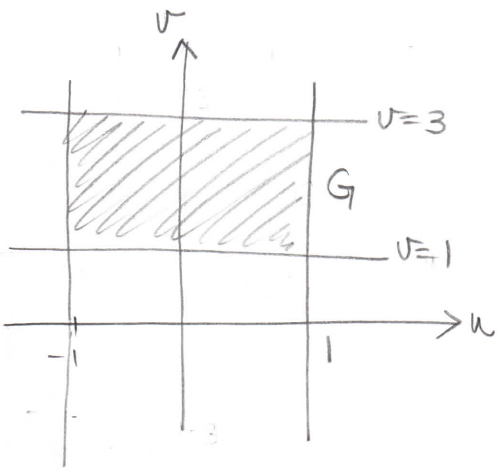
$$= 24\left.\frac{r^4}{4}\right|_0^1 \left.\left(\frac{1}{2}\theta+\frac{1}{4}\sin(2\theta)\right)\right|_0^{2\pi}$$

$$= 6\left(\pi+\frac{1}{4}\sin(4\pi)-0\right)$$

$$= \boxed{6\pi}$$

5. $\iint_R \left(\frac{x-y}{x+y+2}\right)^2 dx\,dy$





$$\begin{cases} u = x-y \\ v = x+y+2 \end{cases}$$

$2x+2 = u+v \Rightarrow x = \frac{1}{2}(u+v)-1$

$2y+2 = -u+v \Rightarrow y = \frac{1}{2}(-u+v)-1$

$$\begin{cases} x = \frac{1}{2}(u+v)-1 \\ y = \frac{1}{2}(-u+v)-1 \end{cases}$$

$$J = \begin{vmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{vmatrix} = \frac{1}{2}.$$

$$\iint_R \left(\frac{x-y}{x+y+2}\right)^2 dx\,dy = \iint_G \left(\frac{u}{v}\right)^2 \cdot \frac{1}{2}\,du\,dv$$

$$= \int_{-1}^{1}\int_{1}^{3} \frac{1}{2}\frac{u^2}{v^2}\,dv\,du$$

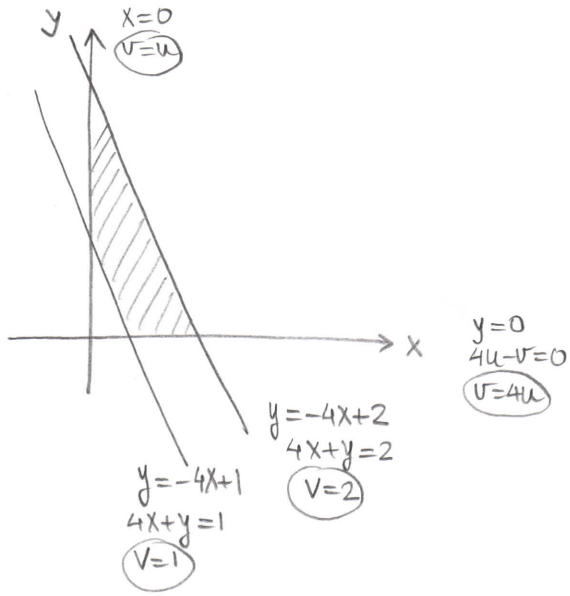
$$= \frac{1}{2}\left(\frac{u^3}{3}\Big|_{-1}^{1}\right)\left(-\frac{1}{v}\right)\Big|_{1}^{3}$$

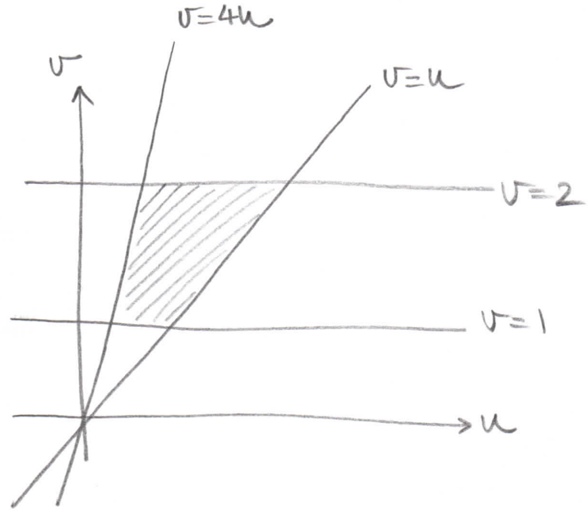
$$= \frac{1}{2}\left(\frac{1}{3}+\frac{1}{3}\right)\left(-\frac{1}{3}+1\right)$$

$$= \cancel{\frac{1}{2}} \cdot \cancel{\frac{2}{3}} \cdot \frac{2}{3} = \boxed{\frac{2}{9}}$$

⑥ $\iint_R e^{\frac{x+y}{4x+y}}\,dA$; $R=\{(x,y): 1\le 4x+y\le 2 ;\ x\ge 0 ;\ y\ge 0\}$

$-4x+1\le y\le -4x+2$





Substitution: $\begin{cases} u = x+y \\ v = 4x+y \end{cases} \Rightarrow \begin{cases} x=\frac{1}{3}(-u+v) \\ y=\frac{1}{3}(4u-v) \end{cases} \Rightarrow J=\begin{vmatrix} -1/3 & 1/3 \\ 4/3 & -1/3 \end{vmatrix} = \frac{1}{9}-\frac{4}{9} = -1/3$

$3x=v-u$

$$\begin{array}{r} 4x+4y=4u \\ 4x+y=v \\ \hline 3y=4u-v \end{array}$$

$$\Rightarrow \iint_R e^{\frac{x+y}{4x+y}}\,dA = \iint_G e^{u/v}\,\frac{1}{3}\,dA = \frac{1}{3}\int_1^2\int_{1/4v}^{v} e^{u/v}\,du\,dv$$

$$=\frac{1}{3}\int_1^2 v\,e^{u/v}\Big|_{u=1/4v}^{u=v}\,dv$$

$$=\frac{1}{3}\int_1^2 v\left(e^1-e^{1/4}\right)dv$$

$$=\frac{1}{3}\left(e-e^{1/4}\right)\frac{v^2}{2}\Big|_1^2$$

$$=\boxed{\frac{1}{2}\left(e-e^{1/4}\right)}$$