Worksheet 16 - Solutions

1

$$\vec{F} = \langle 2xy^3z^4,\ 3x^2y^2z^4,\ 4x^2y^3z^3 \rangle$$
$$= \nabla f$$
$$= \left\langle \frac{\partial f}{\partial x},\ \frac{\partial f}{\partial y},\ \frac{\partial f}{\partial z} \right\rangle$$

$$\frac{\partial f}{\partial x} = 2xy^3z^4 \Rightarrow f(x,y,z) = x^2y^3z^4 + g(y,z)$$

$$\left.\begin{array}{l} \Rightarrow \frac{\partial f}{\partial y} = 3x^2y^2z^4 + \frac{\partial g}{\partial y} \\ \text{But } \frac{\partial f}{\partial y} = 3x^2y^2z^4 \end{array}\right\} \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y,z) = h(z)$$

$$\Rightarrow f(x,y,z) = x^2y^3z^4 + h(z)$$

$$\left.\begin{array}{l} \Rightarrow \frac{\partial f}{\partial z} = 4x^2y^3z^3 + h'(z) \\ \text{But } \frac{\partial f}{\partial z} = 4x^2y^3z^3 \end{array}\right\} \Rightarrow$$

$$\Rightarrow h'(z) = 0 \Rightarrow h(z) = C$$

$$\Rightarrow \boxed{f(x,y,z) = x^2y^3z^4 + C}$$

2

$$\vec{F} = \langle \underbrace{2x\cos(y) - 2z^3}_{\partial f/\partial x},\ \underbrace{3 + 2ye^z - x^2\sin(y)}_{\partial f/\partial y},\ \underbrace{y^2e^z - 6xz^2}_{\partial f/\partial z} \rangle$$

$$\frac{\partial f}{\partial x} = 2x\cos(y) - 2z^3 \Rightarrow f(x,y,z) = x^2\cos(y) - 2z^3x + g(y,z)$$

$$\left.\begin{array}{l} \Rightarrow \frac{\partial f}{\partial y} = -x^2\sin(y) + \frac{\partial g}{\partial y} \\ \text{(given)} = -x^2\sin(y) + 3 + 2ye^z \end{array}\right\} \Rightarrow \frac{\partial g}{\partial y} = 3 + 2ye^z$$

$$\Rightarrow g(y,z) = 3y + y^2e^z + h(z)$$

$$\Rightarrow f(x,y,z) = x^2\cos(y) - 2z^3x + 3y + y^2e^z + h(z)$$

$$\left.\begin{array}{l} \Rightarrow \frac{\partial f}{\partial z} = -6z^2x + y^2e^z + h'(z) \\ \text{(given)} = -6z^2x + y^2e^z \end{array}\right\} \Rightarrow h'(z) = 0 \Rightarrow h(z) = C$$

$$\boxed{f(x,y,z) = x^2\cos(y) - 2z^3x + 3y + y^2e^z + C}$$

(3) $\vec{F}(x,y) = \langle 2x^3y^4 + x \ , \ 2x^4y^3 + y \rangle$

$M = 2x^3y^4 + x$
$N = 2x^4y^3 + y$

(a) $\left.\begin{array}{l} \dfrac{\partial M}{\partial y} = 8x^3y^3 \\ \dfrac{\partial N}{\partial x} = 8x^3y^3 \end{array}\right\} \Rightarrow \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x} \Rightarrow \vec{F}$ is conservative ✓

(b) $\vec{F}(x,y) = \nabla f(x,y)$

$$\frac{\partial f}{\partial x} = 2x^3y^4 + x \Rightarrow f(x,y) = \frac{1}{2}x^4y^4 + \frac{x^2}{2} + g(y)$$

$$\left.\begin{array}{rl} \Rightarrow \dfrac{\partial f}{\partial y} &= 2x^4y^3 + g'(y) \\ \text{(given)} &= 2x^4y^3 + y \end{array}\right\} \Rightarrow g'(y) = y \Rightarrow g(y) = \frac{y^2}{2} + C$$

$$\Rightarrow \boxed{f(x,y) = \frac{1}{2}x^4y^4 + \frac{x^2}{2} + \frac{y^2}{2} + C}$$

(c) $\int_C \vec{F}\cdot d\vec{r}$; $\vec{r}(t) = \left\langle t\cos(\pi t) - 1, \ \sin\left(\frac{\pi t}{2}\right)\right\rangle$; $0 \le t \le 1$

Since $\vec{F}$ is conservative, we can apply FTC for Line Integrals:

$A = \vec{r}(0) = (-1,0)$
$B = \vec{r}(1) = (-2,1)$

$$\begin{aligned} \int_C \vec{F}\cdot d\vec{r} &= \int_A^B \vec{F}\cdot d\vec{r} \\ &= f(B) - f(A) \\ &= f(-2,1) - f(-1,0) \\ &= \left(8 + 2 + \frac{1}{2} - \frac{1}{2}\right) = \boxed{10} \end{aligned}$$

(d) $\int_C \vec{F}\cdot d\vec{r} = \int_C M\,dx + N\,dy = \int_C (2x^3y^4 + x)\,dx + (2x^4y^3 + y)\,dy$

$x = t\cos(\pi t) - 1$; $dx = \left(\cos(\pi t) - t\pi\sin(\pi t)\right)dt$
$y = \sin\left(\frac{\pi t}{2}\right)$; $dy = \frac{\pi}{2}\cos\left(\frac{\pi t}{2}\right)dt$

④ $\vec{F}(x,y,z) = \langle e^x\cos y + yz,\ xz - e^x\sin y,\ xy + z\rangle$

(a) $\text{Curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ e^x\cos y + yz & xz - e^x\sin y & xy+z \end{vmatrix}$

$= \langle x - x,\ -(y-y),\ (z - e^x\sin y) - (-e^x\sin y + z)\rangle = \vec{0}$

(b) $\vec{F}(x,y,z) = \nabla f$

$\frac{\partial f}{\partial x} = e^x\cos y + yz \Rightarrow f(x,y,z) = e^x\cos y + xyz + g(y,z)$

$\left.\begin{aligned} \Rightarrow \frac{\partial f}{\partial y} &= -e^x\sin y + xz + \frac{\partial g}{\partial y} \\ \text{(given)} &= -e^x\sin y + xz \end{aligned}\right\} \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y,z) = h(z)$

$\Rightarrow f(x,y,z) = e^x\cos y + xyz + h(z)$

$\left.\begin{aligned} \Rightarrow \frac{\partial f}{\partial z} &= xy + h'(z) \\ \text{(given)} &= xy + z \end{aligned}\right\} \Rightarrow h'(z) = z \Rightarrow h(z) = \frac{z^2}{2} + C$

$$\boxed{f(x,y,z) = e^x\cos y + xyz + \frac{z^2}{2} + C}$$

(c) $\int_{(0,0,0)}^{(0,\pi,1)} \vec{F}\cdot d\vec{r} = f(0,\pi,1) - f(0,0,0) = \left(-1 + \frac{1}{2}\right) - (1) = \boxed{-\frac{3}{2}}$

(d) $\oint_C \vec{F}\cdot d\vec{r} = \boxed{0}$ for any loop C, since $\vec{F}$ is conservative.

5. $\vec{F}(x,y,z) = \left\langle 2\cos y,\ \frac{1}{y} - 2x\sin y,\ \frac{1}{z} \right\rangle$

$$\frac{\partial f}{\partial x} = 2\cos y \Rightarrow f(x,y,z) = 2x\cos y + g(y,z)$$

$$\left.\begin{aligned} \Rightarrow \frac{\partial f}{\partial y} &= -2x\sin y + \frac{\partial g}{\partial y} \\ &= -2x\sin y + \frac{1}{y} \end{aligned}\right\} \Rightarrow \frac{\partial g}{\partial y} = \frac{1}{y} \Rightarrow g(y,z) = \ln|y| + h(z)$$

$$\Rightarrow f(x,y,z) = 2x\cos y + \ln|y| + h(z)$$

$$\left.\begin{aligned} \Rightarrow \frac{\partial f}{\partial z} &= h'(z) \\ &= \frac{1}{z} \end{aligned}\right\} \Rightarrow h(z) = \ln|z| + C$$

$$\boxed{f = 2x\cos y + \ln|y| + \ln|z| + C}$$

By this point, you may already be able to tell the potential function - I am just writing the rest in detail in case ∃ any confusion

6. $\vec{F}(x,y,z) = \langle -2xy^2\sin(x^2y^2)\sin z + y\cos(xy)\,e^{\sin(xy)}z,$
$-2x^2y\sin(x^2y^2)\sin z + x\cos(xy)\,e^{\sin(xy)}z,$
$\cos(x^2y^2)\cos(z) + e^{\sin(xy)} \rangle.$

Which component function is easiest to integrate?

$$\int \left(\cos(x^2y^2)\cos(z) + e^{\sin(xy)}\right) dz = \cos(x^2y^2)\sin(z) + e^{\sin(xy)}\cdot z + g(x,y)$$

$$\Rightarrow f(x,y,z) = \cos(x^2y^2)\sin z + e^{\sin(xy)}\cdot z + g(x,y)$$

$$\Rightarrow \frac{\partial f}{\partial y} = -2x^2y\sin(x^2y^2)\sin z + x\cos(xy)\,e^{\sin(xy)}\cdot z + \underbrace{\frac{\partial g}{\partial y}}_{=0} \Rightarrow g(x,y) = h(x)$$

$$\Rightarrow f(x,y,z) = \cos(x^2y^2)\sin z + e^{\sin(xy)}\cdot z + h(x)$$

$$\Rightarrow \frac{\partial f}{\partial x} = -2xy^2\sin(x^2y^2)\sin z + y\cos(xy)\,e^{\sin(xy)}\cdot z + \underbrace{h'(x)}_{=0} \Rightarrow h(x) = C$$

$$\boxed{f = \cos(x^2y^2)\sin z + e^{\sin(xy)}\cdot z + C}$$