Worksheet 6 ~ Solutions

1 $\vec{r}(t) = \langle t^2, \sin t, t \rangle$; $3 \le t \le 9$.

$t = e^s \Rightarrow \boxed{\vec{r}_1(s) = \langle e^{2s}, \sin(e^s), e^s \rangle}$; $\boxed{\ln 3 \le s \le \ln 9}$

$s = \ln t \longrightarrow 3 \le t \le 9 \Rightarrow \ln 3 \le \ln t = s \le \ln 9$

2 $\vec{r}(t) = \langle 2t, 1-2t, t \rangle$ Base point : $(0,1,0) \Rightarrow t_0 = 0$

$\vec{r}'(t) = \langle 2, -2, 1 \rangle \Rightarrow |\vec{r}'(t)| = 3 \Rightarrow s(t) = \int_0^t 3\,d\tau = 3t$ $\quad s = 3t \Rightarrow t = s/3$

$\Rightarrow$ Reparametrization : $\boxed{\vec{r}_1(s) = \langle 2s/3, 1 - 2s/3, s/3 \rangle}$

3 $\kappa(t) = \dfrac{1}{|\vec{v}(t)|} \left| \dfrac{d\vec{T}}{dt} \right|$

(a). Show: $\boxed{\kappa(t) = \dfrac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3}}$

$$\vec{v}(t) = |\vec{v}(t)|\,\vec{T}(t) = \frac{ds}{dt}\vec{T}(t)$$

$$\Rightarrow \vec{a}(t) = \frac{d}{dt}\left(\frac{ds}{dt}\vec{T}(t)\right) = \frac{d^2s}{dt^2}\vec{T}(t) + \frac{ds}{dt}\frac{d\vec{T}}{dt} \quad \text{(Product Rule)}$$

$$\Rightarrow \vec{v}(t) \times \vec{a}(t) = \left(\frac{ds}{dt}\vec{T}(t)\right) \times \left(\frac{d^2s}{dt^2}\vec{T}(t) + \frac{ds}{dt}\frac{d\vec{T}}{dt}\right)$$

$$= \left(\frac{ds}{dt}\right)\left(\frac{d^2s}{dt^2}\right)\underbrace{\vec{T}(t) \times \vec{T}(t)}_{0\text{, because } \vec{T} \parallel \vec{T}} + \underbrace{\left(\frac{ds}{dt}\right)^2}_{|\vec{v}(t)|^2} \vec{T}(t) \times \frac{d\vec{T}}{dt}$$

$$\Rightarrow \boxed{\vec{v}(t) \times \vec{a}(t) = |\vec{v}(t)|^2 \left(\vec{T}(t) \times \frac{d\vec{T}}{dt}\right)}$$

$$\Rightarrow |\vec{v}(t) \times \vec{a}(t)| = |\vec{v}(t)|^2 \left|\vec{T}(t) \times \frac{d\vec{T}}{dt}\right|$$

$$= |\vec{v}(t)|^2 \underbrace{|\vec{T}(t)|}_{1} \left|\frac{d\vec{T}}{dt}\right| \underbrace{\sin\theta}_{1}$$

$\theta = \measuredangle$ b/w $\vec{T}$ & $\dfrac{d\vec{T}}{dt} \Rightarrow \theta = \pi/2$

$\vec{T}$ has constant length $\Rightarrow \vec{T} \perp \dfrac{d\vec{T}}{dt}$

$$\Rightarrow \boxed{|\vec{v}(t) \times \vec{a}(t)| = |\vec{v}(t)|^2 \left|\frac{d\vec{T}}{dt}\right|}$$

$$\Rightarrow \left|\frac{d\vec{T}}{dt}\right| = \frac{1}{|\vec{v}(t)|^2}|\vec{v}(t) \times \vec{a}(t)| \Rightarrow \kappa(t) = \frac{1}{|\vec{v}(t)|^3}|\vec{v}(t) \times \vec{a}(t)|$$

(b). $\vec{r}(t) = \langle t, t^2, t^3 \rangle$

$\Rightarrow \vec{v}(t) = \langle 1, 2t, 3t^2 \rangle \quad \Rightarrow \quad |\vec{v}(t)| = \sqrt{1+4t^2+9t^4}$

$\Rightarrow \vec{a}(t) = \langle 0, 2, 6t \rangle$

$$\Rightarrow \vec{v}(t) \times \vec{a}(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = \langle 6t^2, -6t, 2 \rangle$$

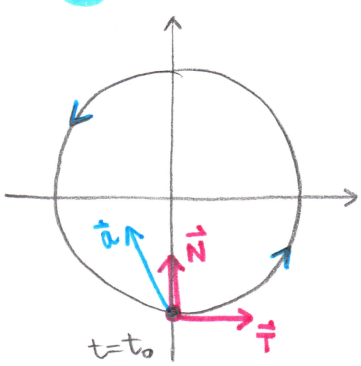
$$\Rightarrow |\vec{v}(t) \times \vec{a}(t)| = \sqrt{36t^4 + 36t^2 + 4} = 2\sqrt{9t^4 + 9t^2 + 1}$$

$$\Rightarrow \boxed{\kappa(t) = \frac{2\sqrt{9t^4+9t^2+1}}{(1+4t^2+9t^4)^{3/2}}}$$

Remark: Using the original formula, you would have to differentiate

$\vec{T}(t) = \dfrac{1}{\sqrt{1+4t^2+9t^4}} \langle 1, 2t, 3t^2 \rangle$ with respect to time, then find $\left|\dfrac{d\vec{T}}{dt}\right|$.

5



Rotating counterclockwise $\Rightarrow$ $\underbrace{\vec{T} = \langle 1, 0 \rangle}_{=\vec{i}}$ $\underbrace{\vec{N} = \langle 0, 1 \rangle}_{=\vec{j}}$ at $t = t_0$

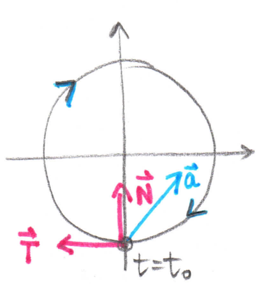
$$\left.\begin{array}{l} |\vec{v}(t_0)| = 30\ ^{m}/_{min} \\ R = 30\,m \Rightarrow \kappa(t_0) = \dfrac{1}{30m} \\ \text{(curvature of a circle)} \end{array}\right\} \Rightarrow a_N(t_0) = \kappa(t_0)|\vec{v}(t_0)|^2 = \frac{1}{30m}(30)^2 \frac{m^2}{min^2} = 30\ ^{m}/_{min^2}$$

$$\boxed{a_N(t_0) = 30\ ^{m}/_{min^2}}$$

$$a_T(t_0) = \frac{d}{dt}|\vec{v}(t)|\bigg|_{t=t_0} = -15\ ^{m}/_{min^2} \qquad \boxed{a_T(t_0) = -15\ ^{m}/_{min^2}}$$

$$\Rightarrow \vec{a}(t_0) = -15\underbrace{\vec{T}}_{\vec{i}} + 30\underbrace{\vec{N}}_{\vec{j}} \Rightarrow \boxed{\vec{a}(t_0) = \langle -15, 30 \rangle}$$

How would the answer change if motion of wheel was clockwise?



$$\left.\begin{array}{l} \vec{N} = \langle 0, 1 \rangle = \vec{j} \\ \vec{T} = \langle -1, 0 \rangle = -\vec{i} \end{array}\right\} \Rightarrow \vec{a}(t_0) = -15\vec{T} + 30\vec{N}$$

$$= -15(-\vec{i}) + 30\vec{j} \Rightarrow \boxed{\vec{a}(t_0) = \langle 15, 30 \rangle}$$

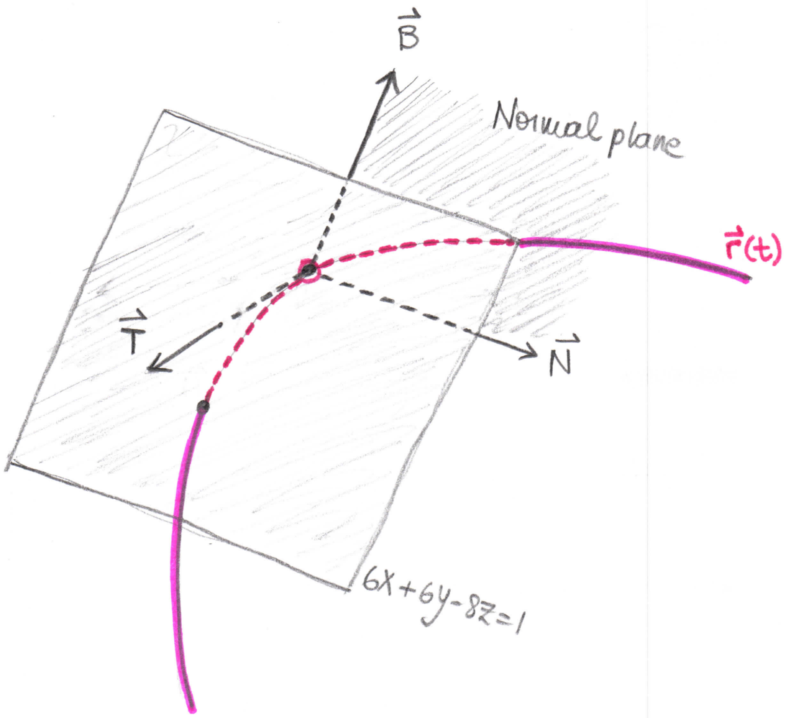
④

(Normal Plane) // $(6x+6y-8z)=1$
$\vec{T} \perp$ (Normal Plane)
$\langle 6,6,-8\rangle \perp [(6x+6y-8z)=1]$
$\Rightarrow$

$\Rightarrow \boxed{\vec{T} \,//\, \langle 6,6,-8\rangle}$

$\Rightarrow \boxed{\vec{T} = k\langle 6,6,-8\rangle}$

for some scalar k



$\vec{r}(t) = \langle t^3, 3t, t^4\rangle \Rightarrow \vec{v}(t) = \langle 3t^2, 3, 4t^3\rangle \Rightarrow |\vec{v}(t)| = \sqrt{9t^4+9+16t^6}$

$\Rightarrow \boxed{\vec{T}(t) = \frac{1}{\sqrt{16t^6+9t^4+9}}\langle 3t^2, 3, 4t^3\rangle} \rightarrow \boxed{= k\langle 6,6,-8\rangle}$

$\Rightarrow \begin{cases} (1) & 3t^2 = 6k\sqrt{16t^6+9t^4+9} \\ (2) & 3 = 6k\sqrt{16t^6+9t^4+9} \\ (3) & 4t^3 = -8k\sqrt{16t^6+9t^4+9} \end{cases}$

$3t^2 = 3 \Rightarrow t^2 = 1 \Rightarrow t = \pm 1$ which one is correct? are they both solutions?

(3): $t=1 \Rightarrow 4 = -8k\sqrt{34} \Rightarrow k = -\frac{1}{2\sqrt{34}}$ ← Can't be! $k \geq 0$! Why? Because of (1) & (2):

(3): $t=-1 \Rightarrow -4 = -8k\sqrt{34} \Rightarrow k = \frac{1}{2\sqrt{34}}$ ✓

$3t^2$ and 3 are both ≥ 0

$\Rightarrow 6k\underbrace{\sqrt{16t^6+9t^4+9}}_{\geq 0} \geq 0 \Rightarrow k \geq 0$

$\Rightarrow t=-1 \Rightarrow \vec{r}(-1) = \langle -1,-3,1\rangle$

$\Rightarrow \boxed{\text{Point: } (-1,-3,1)}$ //