Name: Solutions

March 9^{th}, 2016.
Math 2551; Sections L1, L2, L3.
Georgia Institute of Technology
EXAM 2

I commit to uphold the ideals of honor and integrity by refusing to betray the trust bestowed upon me as a member of the Georgia Tech community. By signing my name below I pledge that I have neither given nor received help on this exam.

Pledged: ______________________

Problem	Possible Score	Earned Score
1	18	
2	18	
3	18	
4	16	
5	15	
6	15	
Total	100	

Remember that you must SHOW YOUR WORK to receive credit!

Good luck!

1. [18 points] Find an equation for the plane tangent to the surface:

$$-8\cos(\pi x) + 5x^2y + 5e^{xz} + 4yz = 18,$$

at the point $P_0(1,1,0)$. Express the equation in the form $Ax + By + Cz = D$.

$$f(x,y,z) = -8\cos(\pi x) + 5x^2y + 5e^{xz} + 4yz$$

$$\nabla f = \langle 8\pi \sin(\pi x) + 10xy + 5ze^{xz},\ 5x^2 + 4z,\ 5xe^{xz} + 4y \rangle$$

12 pts

4pts. each component

$$\nabla f(1,1,0) = \langle 8\pi \sin(\pi) + 10,\ 5,\ 5+4 \rangle$$

$$= \boxed{\langle 10, 5, 9 \rangle}$$

2pts.

Tangent Plane: $10(x-1) + 5(y-1) + 9z = 0$

$$\boxed{10x + 5y + 9z = 15}$$

4pts.

2. [18 points] Find the direction(s) $\mathbf{u}$ for which the directional derivative $D_{\mathbf{u}}f(1,-1) = 0$, where

$$f(x,y) = x^2 - xy + y^2 - y.$$

$\nabla f = \langle 2x-y, -x+2y-1\rangle$ 3 pts. - gradient

$\nabla f(1,-1) = \langle 3,-4\rangle$ 2 pts. - evaluate @(1,-1)

$D_{\vec{u}}f(1,-1) = \nabla f(1,-1)\cdot\vec{u}$ 2 pts. - formula

$\vec{u} = \langle a,b\rangle$ unit vector

$$\begin{cases}\langle 3,-4\rangle\cdot\langle a,b\rangle = 0\\ a^2+b^2=1\end{cases}$$ 3 pts - setup

6 pts. Solving system

$$\begin{cases}3a-4b=0 \Rightarrow a=\frac{4}{3}b\\ a^2+b^2=1\end{cases}$$

$$\frac{16}{9}b^2+b^2=1 \Rightarrow \frac{25}{9}b^2=1 \Rightarrow b^2=\frac{9}{25} \Rightarrow b=\pm\frac{3}{5}$$

$b=\frac{3}{5} \Rightarrow a=\frac{4}{5}$

$b=-\frac{3}{5} \Rightarrow a=-\frac{4}{5}$

$\Rightarrow$ Directions: $\boxed{\langle 4/5, 3/5\rangle \ \& \ \langle -4/5, -3/5\rangle}$ 2 pts. final answer

3. [18 points] Find all the critical points of the function

$$f(x,y) = 8x^2 + 4x^2y + y^2 - 7,$$

and classify each one as a local maximum, a local minimum, or a saddle point.

$$\nabla f = \langle 16x + 8xy,\ 4x^2 + 2y \rangle$$ 2 pts. gradient

2 pts. Setup

$$\begin{cases} 16x + 8xy = 0 \\ 4x^2 + 2y = 0 \end{cases} \qquad \begin{cases} 8x(2+y) = 0 \\ 2x^2 + y = 0 \end{cases} \Rightarrow x = 0 \ \underline{\text{OR}} \ y = -2$$

$$x = 0 \Rightarrow y = 0$$

$$y = -2 \Rightarrow 2x^2 - 2 = 0 \Rightarrow x = \pm 1$$

$\Rightarrow$ Critical points: $(1,-2)$; $(-1,-2)$; $(0,0)$ 6 pts. 2pts. each critical pt.

$$\Delta_f = f_{xx} f_{yy} - f_{xy}^2$$

$$= 2(16 + 8y) - 16x^2$$

$$= 32 + 16y - 16x^2$$

$$\left.\begin{aligned} f_{xx} &= 16 + 8y \\ f_{yy} &= 2 \\ f_{xy} &= 8x \end{aligned}\right\}$$ 2 pts.

$$\left.\begin{aligned} \Delta_f(0,0) &= 32 > 0 \\ f_{xx}(0,0) &= 16 > 0 \end{aligned}\right\} \Rightarrow (0,0) \text{ is a } \underline{\text{local min}}$$

$$\Delta_f(-1,-2) = 32 - 32 - 16 < 0 \Rightarrow (-1,-2) \text{ is a } \underline{\text{saddle point}}$$

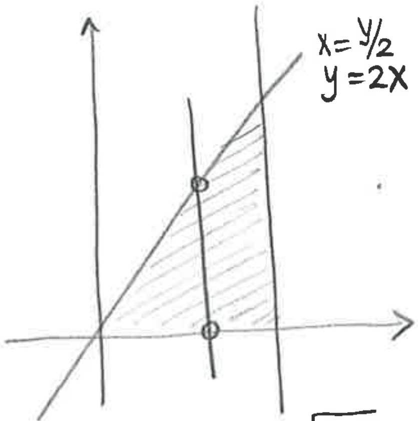
$$\Delta_f(1,-2) = 32 - 32 - 16 < 0 \Rightarrow (1,-2) \text{ is a } \underline{\text{saddle point}}$$

6 pts. 2pts. each

4. [16 points] Evaluate the integral:

$$\int_0^{2\sqrt{\ln 6}} \int_{y/2}^{\sqrt{\ln 6}} e^{x^2}\, dx\, dy.$$

2pts. intent $\rightarrow$ Switch the order of integration:



Take vertical cross-sections

$$\int_0^{\sqrt{\ln 6}} \int_0^{2x} e^{x^2}\, dy\, dx \quad \leftarrow \text{8pts.}$$

$$= \int_0^{\sqrt{\ln 6}} y e^{x^2} \Big|_{y=0}^{y=2x} dx \qquad \text{2pts.}$$

$$= \int_0^{\sqrt{\ln 6}} 2x e^{x^2} dx \qquad \text{2pts.}$$

$$= e^{x^2} \Big|_0^{\sqrt{\ln 6}} \qquad \text{1pt.}$$

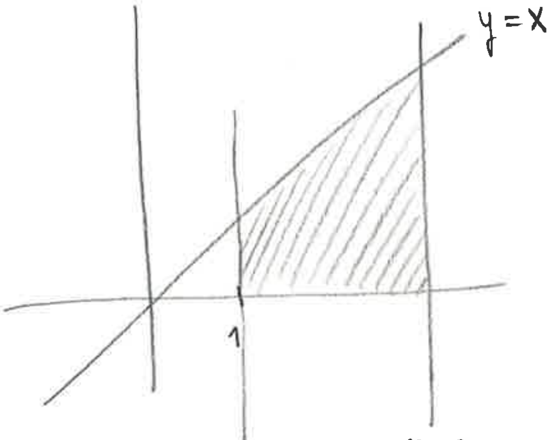
$$= e^{\ln 6} - 1 = \boxed{5} \qquad \text{1pt.}$$

5. [15 points] Evaluate the integral:

$$\int_1^2 \int_0^x \frac{1}{(x^2+y^2)^{3/2}}\,dy\,dx.$$

Change to polar: 2pts. intent

$$\int_0^{\pi/4} \int_{1/\cos\theta}^{2/\cos\theta} \frac{1}{(r^2)^{3/2}}\, r\,dr\,d\theta$$ 8pts.



x = 1

$r\cos\theta = 1$

$r = 1/\cos\theta$

x = 2

$r\cos\theta = 2$

$r = 2/\cos\theta$

$$= \int_0^{\pi/4} \int_{1/\cos\theta}^{2/\cos\theta} \frac{1}{r^2}\,dr\,d\theta$$

$$= \int_0^{\pi/4} -\frac{1}{r}\Big|_{1/\cos\theta}^{2/\cos\theta} d\theta$$ 2pts.

$$= \int_0^{\pi/4} \left(-\frac{\cos\theta}{2} + \cos\theta\right) d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} \cos\theta\,d\theta$$ 2pts.

$$= \frac{1}{2}\sin\theta\Big|_0^{\pi/4}$$

$$= \frac{1}{2}\cdot\frac{\sqrt{2}}{2}$$

$$= \boxed{\frac{\sqrt{2}}{4}}$$ 1pt.

6. [15 points] Find the point(s) on the surface $z^2 = xy + 4$ which are closest to the origin.

Minimize $f(x,y,z) = x^2+y^2+z^2$

Subject to $g(x,y,z) = z^2 - xy - 4 = 0$

3 pts. Setup

Lagrange multipliers:

$\nabla f = \langle 2x, 2y, 2z \rangle$

$\nabla g = \langle -y, -x, 2z \rangle$

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 0 \end{cases}$$

2pts.

8 pts. Solving system

$$\begin{cases} 2x = -\lambda y \\ 2y = -\lambda x \\ 2z = 2\lambda z \longrightarrow z(\lambda - 1) = 0 \Rightarrow z = 0 \text{ or } \lambda = 1 \\ z^2 = xy + 4 \end{cases}$$

- $z = 0$: $\begin{cases} 2x = -\lambda y \Rightarrow -\lambda = \frac{2x}{y} \\ 2y = -\lambda x \Rightarrow -\lambda = \frac{2y}{x} \\ xy + 4 = 0 \end{cases} \Rightarrow \frac{2x}{y} = \frac{2y}{x} \Rightarrow 2x^2 = 2y^2 \Rightarrow x = \pm y$

  $x = y \Rightarrow x^2 + 4 = 0$ impossible

  $x = -y \Rightarrow -x^2 + 4 = 0 \Rightarrow x = \pm 2$ $\boxed{(2,-2,0) \ \& \ (-2,2,0)}$

- $\lambda = 1$: $\begin{cases} 2x = -y \Rightarrow y = -2x \\ 2y = -x \Rightarrow -4x = -x \Rightarrow x = 0 \Rightarrow y = 0 \\ z^2 = xy + 4 \Rightarrow z^2 = 4 \Rightarrow z = \pm 2 \end{cases}$ $\boxed{(0,0,-2) \ \& \ (0,0,2)}$

$\Rightarrow$ Points closest to the origin $\boxed{(0,0,-2) \text{ and } (0,0,2)}$ 2pts.