Homework 1

Separable Equations

1. $\frac{dy}{dx} = \sin(5x)$

$dy = \sin(5x)\,dx$

Integrate both sides : $\int dy = \int \sin(5x)\,dx$

$y = -\frac{1}{5}\cos(5x) + C$

$$\boxed{y = -\frac{1}{5}\cos(5x) + C}$$

2. $xy' = 4y$

$x\frac{dy}{dx} = 4y \Rightarrow \frac{1}{4y}dy = \frac{1}{x}dx$

$\Rightarrow \frac{1}{4}\ln|y| = \ln|x| + C$

$\Rightarrow e^{\frac{1}{4}\ln|y|} = e^{\ln|x|+C}$

$\Rightarrow |y|^{1/4} = |x|e^{C} \Rightarrow |y| = C|x|^4$

$\Rightarrow y = \pm C|x|^4$

$$\boxed{y = Cx^4}$$

Remark : we divided by y
Is $y=0$ a valid solution? Yes
Did we lose a solution? No
because $y=0$ is represented in $y = Cx^4$

3. $(4y + yx^2)dy - (2x + xy^2)dx = 0$

$y(4+x^2)dy = x(2+y^2)dx$

$\frac{y}{2+y^2}dy = \frac{x}{4+x^2}dx$

$\frac{1}{2}\ln|2+y^2| = \frac{1}{2}\ln|4+x^2| + C$

$\ln(2+y^2) = \ln(4+x^2) + C$

$e^{\ln(2+y^2)} = e^{\ln(4+x^2)+C}$

$2+y^2 = (4+x^2)e^C$

$$\boxed{2+y^2 = c(4+x^2)}$$

Any issues when dividing?
No, because both $(2+y^2)$, $(4+x^2) > 0$

4 $\dfrac{dy}{dx} = \dfrac{xy+3x-y-3}{xy-2x+4y-8} = \dfrac{(x-1)(y+3)}{(x+4)(y-2)}$

$$\frac{y-2}{y+3}\,dy = \frac{x-1}{x+4}\,dx$$

$$\int \frac{y-2}{y+3}\,dy = \int \frac{(y+3)-5}{y+3}\,dy = \int\left(1-\frac{5}{y+3}\right)dy = y-5\ln|y+3|+C$$

$$\int \frac{x-1}{x+4}\,dx = \int \frac{(x+4)-5}{x+4}\,dx = \int\left(1-\frac{5}{x+4}\right)dx = x-5\ln|x+4|+C$$

$$\boxed{y-5\ln|y+3| = x-5\ln|x+4|+C}$$ $\leftarrow$ *perfectly good answer*

or you can simplify further:

$$(y-x)+C = 5\ln|y+3| - 5\ln|x+4|$$
$$= 5\ln\left|\frac{y+3}{x+4}\right|$$
$$\Rightarrow e^{(y-x)+C} = \left|\frac{y+3}{x+4}\right|^5$$
$$\Rightarrow Ce^{(y-x)} = \pm\left(\frac{y+3}{x+4}\right)^5 \Rightarrow \boxed{e^{y-x} = C\left(\frac{y+3}{x+4}\right)^5}$$

5 $y\,dy = 4x\sqrt{y^2+1}\,dx$; $y(0)=1$

$$\int \frac{y}{\sqrt{y^2+1}}\,dy = \int 4x\,dx$$

$$\sqrt{y^2+1} = 2x^2+C$$

$x=0,\ y=1$: $\sqrt{2}=C \Rightarrow \boxed{\sqrt{y^2+1} = 2x^2+\sqrt{2}}$

⑥ $\frac{dx}{dy} = 4(x^2+1)$; $x(\pi/4) = 1.$

$$\int \frac{1}{x^2+1}\,dx = \int 4\,dy \Rightarrow \arctan(x) = 4y + C$$

$$x = 1 ;\ y = \pi/4 \Rightarrow \arctan(1) = \pi + C$$

$$\pi/4 = \pi + C \Rightarrow C = -3\pi/4$$

$$\Rightarrow \arctan(x) = 4y - 3\pi/4$$

or $\boxed{x = \tan(4y - 3\pi/4)}$

⑦ $x^2 y' = y - xy$; $y(-1) = -1.$

$$x^2 \frac{dy}{dx} = y(1-x)$$

$$\int \frac{1}{y}\,dy = \int \frac{1-x}{x^2}\,dx$$

$$\ln|y| = -\frac{1}{x} - \ln|x| + C$$

$$\ln|xy| = -\frac{1}{x} + C$$

$$|xy| = e^{-\frac{1}{x}+C} \Rightarrow \boxed{xy = ce^{-\frac{1}{x}}}$$

$$x = y = -1 \Rightarrow 1 = ce \Rightarrow c = e^{-1} \Rightarrow \boxed{xy = e^{-\frac{1}{x}-1}}$$

⑧ $e^y \sin(2x)\,dx + \cos x\,(e^{2y} - y)\,dy = 0$

$$e^y \sin(2x)\,dx = \cos x\,(y - e^{2y})\,dy$$

$$\int \frac{\sin(2x)}{\cos x}\,dx = \int \frac{y - e^{2y}}{e^y}\,dy$$

$$\int \frac{2\sin x \cos x}{\cos x}\,dx = -2\cos x + C$$

$$\int ye^{-y} - e^y\,dy = -\int y(e^{-y})'\,dy - e^y + C$$

$$= -ye^{-y} + \int e^{-y}\,dy - e^{+y} + C$$

$$= -ye^{-y} - e^{-y} - e^y + C$$

$$\Rightarrow \boxed{ye^{-y} + e^{-y} + e^y = 2\cos x + C}$$

9. $\frac{dy}{dx} = (x+y+1)^2$

$u = x+y+1 \Rightarrow \frac{du}{dx} = 1 + \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{du}{dx} - 1$

$\frac{du}{dx} - 1 = u^2 \Rightarrow \int \frac{1}{1+u^2}\,du = \int dx$

$\Rightarrow \arctan(u) = x + c$

$\Rightarrow u = \tan(x+c)$

$\Rightarrow x+y+1 = \tan(x+c) \Rightarrow \boxed{y = \tan(x+c) - x - 1}$

10. $\frac{dy}{dx} = 1 + e^{y-x+5}$

$u = y-x+5 \Rightarrow \frac{du}{dx} = \frac{dy}{dx} - 1$

$\frac{du}{dx} + \cancel{1} = \cancel{1} + e^{u} \Rightarrow \int e^{-u}\,du = \int dx \Rightarrow -e^{-u} = x + c$

$\Rightarrow e^{-u} = c - x \Rightarrow -u = \ln(c-x)$

$\Rightarrow x - y - 5 = \ln(c-x)$

$\Rightarrow \boxed{y = x - 5 - \ln(c-x)}$

11. $\frac{dy}{dx} = 2 + \sqrt{y-2x+3}$

$u = y-2x+3 \Rightarrow \frac{du}{dx} = \frac{dy}{dx} - 2$

$\frac{du}{dx} + 2 = 2 + \sqrt{u} \Rightarrow \int \frac{1}{\sqrt{u}}\,du = \int dx \Rightarrow 2\sqrt{u} = x + C$

$\Rightarrow \sqrt{u} = \frac{1}{2}x + C \Rightarrow u = \left(\frac{1}{2}x + c\right)^2$

$\Rightarrow y - 2x + 3 = \left(\frac{1}{2}x + c\right)^2$

$\Rightarrow \boxed{y = 2x - 3 + \left(\frac{1}{2}x + c\right)^2}$

or: $u = \left(\frac{1}{2}x + c\right)^2 = \left(\frac{x+2c}{2}\right)^2$

relabel: $u = \frac{(x+c)^2}{4} \Rightarrow 4u = (x+c)^2$

$\Rightarrow \boxed{4(y - 2x + 3) = (x+c)^2}$

Linear First Order ODEs - Integrating Factors

① $x \frac{dy}{dx} - 4y = x^6 e^x$

Standard Form: $\frac{dy}{dx} - \frac{4}{x} y = x^5 e^x$

$$p(x) = -\frac{4}{x}$$

$$\int p(x)\,dx = -4\ln|x| + c \Rightarrow \mu(x) = e^{-4\ln|x|} = \frac{1}{x^4}$$

Multiply Equation by $\frac{1}{x^4}$:

$$\frac{1}{x^4}\frac{dy}{dx} - \frac{4}{x^5} y = xe^x$$

$$\frac{d}{dx}\left[\frac{1}{x^4} y\right] = xe^x \Rightarrow \frac{1}{x^4} y = \int xe^x\,dx = xe^x - e^x + c$$

$$\Rightarrow \boxed{y = x^5 e^x - x^4 e^x + cx^4}$$

Interval of validity: $(-\infty, \infty)$

② $\frac{dy}{dx} + 2xy = x \ ; \ y(0) = -3$

$$p(x) = 2x \ ; \ \int 2x\,dx = x^2 + c \ ; \ \mu(x) = e^{x^2}$$

$$e^{x^2}\frac{dy}{dx} + 2xe^{x^2} y = xe^{x^2} \Rightarrow \frac{d}{dx}\left(e^{x^2} y\right) = xe^{x^2}$$

$$\Rightarrow e^{x^2} y = \int xe^{x^2}\,dx = \frac{1}{2}e^{x^2} + c$$

$$\Rightarrow \boxed{y = \frac{1}{2} + ce^{-x^2}}$$

$$x = 0,\ y = -3 \Rightarrow -3 = \frac{1}{2} + c \Rightarrow c = -7/2$$

$$\boxed{y = \frac{1}{2} - \frac{7}{2}e^{-x^2}}$$

Interval of validity: $(-\infty, \infty)$

③ $y' + 3x^2 y = x^2$

$p(x) = 3x^2 \Rightarrow \mu(x) = e^{x^3}$

$$\frac{d}{dx}\left(e^{x^3}y\right) = x^2 e^{x^3} \Rightarrow e^{x^3}y = \frac{1}{3}e^{x^3} + c \Rightarrow \boxed{y = \frac{1}{3} + ce^{-x^3}}$$

Interval of validity: $(-\infty, \infty)$.

④ $\frac{dy}{dx} + 5y = 20$; $y(0) = 2$.

$p(x) = 5 \Rightarrow \mu(x) = e^{5x}$

$$\frac{d}{dx}\left(e^{5x}y\right) = 20e^{5x} \Rightarrow e^{5x}y = 4e^{5x} + c \Rightarrow y = 4 + ce^{-5x}$$

$$x=0, y=2 \Rightarrow 2 = 4 + c \Rightarrow \boxed{y = 4 - 2e^{-5x}} \quad x \in (-\infty, \infty).$$

⑤ $x\,dy = (x\sin x - y)\,dx$

$$\frac{dy}{dx} = \sin x - \frac{1}{x}y$$

$$\frac{dy}{dx} + \frac{1}{x}y = \sin x \quad ; \quad p(x) = \frac{1}{x} ; \quad \int p(x)\,dx = \ln|x| + c$$

Take $x \in (0, \infty)$. Then $\mu(x) = x$, so

$$\frac{d}{dx}(xy) = x\sin x$$

$$\Rightarrow xy = \int x\sin x\,dx$$

$$= -x\cos x + \sin x + c$$

$$\Rightarrow \boxed{y = -\cos x + \frac{1}{x}\sin x + \frac{c}{x}} \quad x \in (0, \infty)$$

Remark:

The expression of the solution is the same on $(-\infty, 0)$.

⑥ $(1+e^x)\frac{dy}{dx}+e^x y=0$

$$\frac{dy}{dx}+\frac{e^x}{1+e^x}y=0$$

$$p(x)=\frac{e^x}{1+e^x} \Rightarrow \int p(x)\,dx=\ln(1+e^x)+C$$

$$\Rightarrow \mu(x)=1+e^x$$

$$\Rightarrow \frac{d}{dx}\left[(1+e^x)y\right]=0 \Rightarrow (1+e^x)y=C$$

$$\Rightarrow \boxed{y=\frac{C}{1+e^x}} \qquad x\in(-\infty,\infty)$$

⑦ $x\frac{dy}{dx}+4y=x^3-x$

$$\frac{dy}{dx}+\frac{4}{x}y=x^2-1$$

$$p(x)=\frac{4}{x}\ ;\ \mu(x)=x^4\ ;\ \frac{d}{dx}(x^4y)=x^4(x^2-1)$$

$$x^4y=\frac{1}{7}x^7-\frac{1}{5}x^5+C$$

$$\boxed{y=\frac{1}{7}x^3-\frac{1}{5}x+Cx^{-4}} \qquad x\in(0,\infty).$$

⑧ $(x+1)\frac{dy}{dx}+y=\ln x\ ;\ y(1)=10$

$$\frac{d}{dx}\left[(x+1)y\right]=\ln x$$

$$(x+1)y=\int \ln x\,dx$$

$$=x\ln x-x+C$$

$x=1;\ y=10$

$$\Rightarrow 20=-1+C \Rightarrow C=21$$

$$\boxed{(x+1)y=x\ln x-x+21}$$

$x\in(0,\infty).$

⑨ $(x+2)^2\frac{dy}{dx}=5-8y-4xy$

$$(x+2)^2\frac{dy}{dx}+4y(x+2)=5$$

$$\frac{dy}{dx}+\frac{4}{x+2}y=\frac{5}{(x+2)^2}$$

$$p(x)=\frac{4}{x+2}\ ;\ \int p(x)\,dx=4\ln|x+2|+C$$

Take $x\in(-2,\infty)$: $\mu(x)=e^{4\ln(x+2)}=(x+2)^4$

$$\Rightarrow \frac{d}{dx}\left[(x+2)^4y\right]=5(x+2)^2$$

$$\Rightarrow (x+2)^4y=\frac{5}{3}(x+2)^3+C$$

$$\Rightarrow \boxed{y=\frac{5}{3}(x+2)^{-1}+C(x+2)^{-4}} \qquad x\in(-2,\infty).$$

(10) $\dfrac{dy}{dx} + y = \dfrac{1-e^{-2x}}{e^{x}+e^{-x}}$

$p(x) = 1; \quad \mu(x) = e^{x}$

$\dfrac{d}{dx}\left(e^{x}y\right) = \dfrac{e^{x}-e^{-x}}{e^{x}+e^{-x}}$

$e^{x}y = \displaystyle\int \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}\,dx$

$= \ln(e^{x}+e^{-x}) + C$

$$\boxed{y = e^{-x}\ln(e^{x}+e^{-x}) + Ce^{-x}}$$

$x \in (-\infty, \infty).$

(11) $x^2y' + x(x+2)y = e^{x}$

$y' + \dfrac{x+2}{x}y = \dfrac{1}{x^2}e^{x}$

$p(x) = \dfrac{x+2}{x}; \quad \displaystyle\int p(x)\,dx = \int\left(1+\frac{2}{x}\right)dx$

$= x + 2\ln(x) \quad$ on $(0,\infty)$

$\Rightarrow \mu(x) = e^{x+2\ln(x)} = x^2e^{x}$

$\dfrac{d}{dx}(x^2e^{x}y) = e^{2x} \Rightarrow x^2e^{x}y = \dfrac{1}{2}e^{2x} + C$

$$\boxed{y = \frac{1}{2x^2}e^{x} + \frac{C}{x^2}e^{-x}}$$

$x \in (0, \infty)$

(12) $x\dfrac{dy}{dx} + (3x+1)y = e^{-3x}$

$\dfrac{dy}{dx} + \left(3+\dfrac{1}{x}\right)y = \dfrac{1}{x}e^{-3x}$

$p(x) = 3 + \dfrac{1}{x}; \quad \displaystyle\int p(x)\,dx = 3x + \ln(x)$ on $(0, \infty)$

$\mu(x) = xe^{3x}$

$\dfrac{d}{dx}\left(xe^{3x}y\right) = 1 \Rightarrow xe^{3x}y = x + C$

$$\boxed{y = e^{-3x} + \frac{C}{x}e^{-3x}}$$

$x \in (0, \infty).$

(13) $x(x-2)y' + 2y = 0; \quad y(3) = 6$

$y' + \dfrac{2}{x(x-2)}y = 0$

$p(x) = \dfrac{2}{x(x-2)} = \dfrac{x-(x-2)}{x(x-2)} = \dfrac{1}{x-2} - \dfrac{1}{x}$

Take $x \in (2,\infty)$ $\rightarrow$ look at initial condition $(x=3, y=6)$

$\displaystyle\int p(x)\,dx = \ln|x-2| - \ln|x|$

$= \ln(x-2) - \ln x = \ln\left(\dfrac{x-2}{x}\right)$

$\Rightarrow \mu(x) = \dfrac{x-2}{x}$

$\Rightarrow \dfrac{d}{dx}\left(\dfrac{x-2}{x}y\right) = 0$

$\Rightarrow \dfrac{x-2}{x}y = C \Rightarrow y = \dfrac{Cx}{x-2}$

$x = 3, y = 6 \Rightarrow 6 = 3C \Rightarrow C = 2$

$$\boxed{y = \frac{2x}{x-2}; \; x \in (2,\infty)}$$

(14) $\frac{dy}{dx} = \frac{y}{y-x}$; $y(5)=2$

Remark : This is not separable, and not linear in y

But look at the reciprocal :

$$\frac{dx}{dy} = \frac{y-x}{y} \Rightarrow \boxed{\frac{dx}{dy} + \frac{1}{y}x = 1} \quad (*)$$

↪ This is linear in x

So apply integrating factors :

$$p(y) = \frac{1}{y} \Rightarrow \int p(y)\,dy = \ln(y)$$

↪ Take $y \in (0,\infty)$

$$\Rightarrow \mu(y) = y$$

$\Rightarrow$ (*) becomes : $y\frac{dx}{dy} + x = y$

$$\frac{d}{dy}(yx) = y$$

$$yx = \int y\,dy = \frac{1}{2}y^2 + C$$

$$\boxed{x = \frac{1}{2}y + \frac{C}{y}}$$

I.C. : $x=5, y=2$: $5 = 1 + \frac{C}{2} \Rightarrow C=8$

$$\boxed{x = \frac{1}{2}y + \frac{8}{y}} \quad y \in (0,\infty).$$

15 $(x+4y^2)dy + 2y\,dx = 0$

Bring to standard form: $(x+4y^2)\dfrac{dy}{dx} + 2y = 0$

$$\frac{dy}{dx} + \frac{2y}{x+4y^2} = 0$$

not linear in y!

So try the same reciprocal trick:

$$\frac{dy}{dx} = -\frac{2y}{x+4y^2} \Rightarrow \frac{dx}{dy} = -\frac{x+4y^2}{2y} = -\frac{1}{2y}x - 2y$$

$$\Rightarrow \boxed{\frac{dx}{dy} + \frac{1}{2y}x = -2y}$$ linear in x ✓

$p(y) = \frac{1}{2y}$; $\int p(y)\,dy = \frac{1}{2}\ln y \Rightarrow \mu(y) = \sqrt{y}$

$y \in (0,\infty)$

$$\Rightarrow \frac{d}{dy}\left(\sqrt{y}\cdot x\right) = -2y\sqrt{y} \Rightarrow \sqrt{y}\cdot x = -2\int y^{3/2}dy$$

$$= -2\cdot\frac{2}{5}y^{5/2} + C$$

$$\Rightarrow \boxed{x = -\frac{4}{5}y^2 + \frac{C}{\sqrt{y}}} \quad y \in (0,\infty).$$

16 $y\,dx + (xy + 2x - ye^y)\,dy = 0$

this guy should already tell you that this will not be linear in y.

$$y\frac{dx}{dy} + xy + 2x - ye^y = 0$$

$$\frac{dx}{dy} + \frac{y+2}{y}x = e^y$$

$$p(y) = \frac{y+2}{y} = 1 + \frac{2}{y} \Rightarrow \int p(y)\,dy = y + 2\ln(y)$$

$y \in (0,\infty)$

$$\Rightarrow \mu(y) = y^2 e^y$$

$$\frac{d}{dy}\left(y^2 e^y x\right) = y^2 e^{2y}$$

$$y^2 e^y x = \int y^2 e^{2y}\,dy$$

$$= \frac{1}{2}\int y^2 (e^{2y})'\,dy = \frac{1}{2}y^2 e^{2y} - \int y e^{2y}\,dy$$

$$= \frac{1}{2}y^2 e^{2y} - \frac{1}{2}ye^{2y} + \frac{1}{4}e^{2y} + C$$

$$\boxed{x = \frac{1}{2}e^y - \frac{1}{2y}e^y + \frac{1}{4y^2}e^y + \frac{C}{y^2}e^{-y}} \quad y \in (0,\infty).$$