Daily Laplace - 4/1 - actually a Variation of Parameters!

Solve: $\boxed{y'' + 4y = \tan(2t)}$ for $y(t)$

Sol: Char. Eqn.: $m^2 + 4 = 0$, $m = \pm 2i$ $\Rightarrow$ Complementary Solution:

$$y_c = c_1 \underbrace{\sin(2t)}_{y_1} + c_2 \underbrace{\cos(2t)}_{y_2}$$

$$W = \begin{vmatrix} \sin(2t) & \cos(2t) \\ 2\cos(2t) & -2\sin(2t) \end{vmatrix} = -2\sin^2(2t) - 2\cos^2(2t) = \boxed{-2}$$

$$W_1 = \begin{vmatrix} 0 & \cos(2t) \\ \tan(2t) & -2\sin(2t) \end{vmatrix} = -\tan(2t)\cdot\cos(2t) = -\sin(2t)$$

$$\Rightarrow u_1' = \frac{1}{2}\sin(2t) \Rightarrow u_1 = \frac{-1}{4}\cos(2t)$$

$$W_2 = \begin{vmatrix} \sin(2t) & 0 \\ 2\cos(2t) & \tan(2t) \end{vmatrix} = \frac{\sin^2(2t)}{\cos(2t)}$$

$$\Rightarrow u_2' = -\frac{1}{2}\frac{\sin^2(2t)}{\cos(2t)} \Rightarrow u_2 = \frac{-1}{2}\int \frac{\sin^2(2t)}{\cos(2t)}\,dt$$

$$= \frac{-1}{2}\int \frac{1-\cos^2(2t)}{\cos(2t)}\,dt$$

$$= \frac{-1}{2}\int \sec(2t)\,dt + \frac{1}{2}\int \cos(2t)\,dt$$

$$= \frac{-1}{2}\ln\left|\sec(2t) + \tan(2t)\right| + \frac{1}{4}\sin(2t)$$

$\Rightarrow$ Particular Solution:

$$\Rightarrow y_p = u_1 y_1 + u_2 y_2$$

$$= -\frac{1}{4}\sin(2t)\cos(2t) + \frac{1}{4}\sin(2t)\cos(2t) - \frac{1}{2}\cos(2t)\ln\left|\sec(2t) + \tan(2t)\right|$$

General Solution:

$$y = c_1\sin(2t) + c_2\cos(2t) - \frac{1}{2}\cos(2t)\ln\left|\sec(2t) + \tan(2t)\right|$$