Daily ODE: 2/10/2022

Solve for $y(x)$:
(Implicit Solution is fine)

$$(x+xy^2)dx+(2y+yx^2)dy=0$$

Solution: Exact? $M=x+xy^2$ $\quad M_y=2xy$
$N=2y+yx^2$ $\quad N_x=2xy$ Yes!

Potential: $\begin{cases}\dfrac{\partial f}{\partial x}=x+xy^2 \\ \dfrac{\partial f}{\partial y}=2y+yx^2\end{cases}$ $\Rightarrow f=\dfrac{x^2}{2}+\dfrac{x^2}{2}y^2+g(y)$

$$\frac{\partial f}{\partial y}=x^2y+\underbrace{g'(y)}_{=2y} \Rightarrow \text{Take } g(y)=y^2$$

$$f(x,y)=\frac{x^2}{2}+\frac{x^2}{2}y^2+y^2$$

Sol: $\boxed{\dfrac{x^2}{2}+\dfrac{x^2y^2}{2}+y^2=C}$

OR

Factor? $x(1+y^2)dx+y(2+x^2)dy=0$ Separable!

$$x(1+y^2)dx=-y(2+x^2)dy$$

$$\int\frac{x}{2+x^2}dx=-\int\frac{y}{1+y^2}dy$$

$$\frac{1}{2}\ln(2+x^2)=-\frac{1}{2}\ln(1+y^2)+C \qquad \cdot 2$$

$$\ln(2+x^2)=-\ln(1+y^2)+C$$

$$e \qquad e$$

$$2+x^2=\frac{1}{1+y^2}\cdot e^C$$

$$\boxed{2+x^2=\frac{1}{1+y^2}\cdot C}$$

Check: Differentiate (dx) both sides of $(2+x^2) = \frac{c}{1+y^2}$:

$$\cancel{2}x = \frac{-c}{(1+y^2)^2} \cdot \cancel{2}y \cdot \frac{dy}{dx} \quad \text{(implicit differentiation)}$$

$$\Rightarrow (1+y^2)x = \frac{-cy}{(1+y^2)} \frac{dy}{dx} \qquad \text{BUT } \frac{c}{1+y^2} = (2+x^2)$$

$$= -y \cdot (2+x^2) \frac{dy}{dx} \Rightarrow (1+y^2)x\,dx = -y(2+x^2)dy$$

$$\Rightarrow (1+y^2)x\,dx + y(2+x^2)dy = 0$$

YES

Side note: The two solutions are equivalent!

$$2+x^2 = \frac{1}{1+y^2} \cdot c$$

$$(2+x^2)(1+y^2) = c$$

$$2 + 2y^2 + x^2 + x^2y^2 = c$$

$$2y^2 + x^2 + x^2y^2 = \underbrace{c-2}_{\text{"}c\text{"}}$$

$$2y^2 + x^2 + x^2y^2 = c \qquad \Big| \cdot \frac{1}{2}$$

$$\boxed{y^2 + \frac{x^2}{2} + \frac{x^2y^2}{2} = c}$$ (the other solution)