Daily ODE - 2/13/2022

Find $y(x)$, for $x \in (0, \infty)$: $\quad \frac{1}{x}\,dy = \left(x\cos(x) + \frac{2y}{x^2}\right)dx$

(either implicit or explicit solution is fine).

Solution: Separable? No (the y in $\frac{2y}{x^2}$ cannot be separated from dx).

Exact? $\quad \underbrace{\left(x\cos(x) + \frac{2y}{x^2}\right)}_{M}dx - \underbrace{\frac{1}{x}}_{N}dy = 0$

$$M_y = \frac{2}{x^2}; \quad N_x = \frac{1}{x^2} \qquad \text{No}$$

Linear? $\quad -\frac{1}{x}\frac{dy}{dx} + x\cos(x) + \frac{2}{x^2}y = 0 \quad$ yes

Standard Form: $\boxed{\frac{dy}{dx} - \frac{2}{x}y = x^2\cos(x)}$ $(*)$

$$p(x) = \frac{-2}{x} \Rightarrow \int p(x)dx = -2\ln|x| = -2\ln(x) \text{ b/c } x>0$$

Integrating Factor: $\mu(x) = e^{-2\ln(x)} = \left(e^{\ln(x)}\right)^{-2} = x^{-2} = \frac{1}{x^2}$

Multiply $(*)$ by $\mu(x) = \frac{1}{x^2}$:

$$\underbrace{\frac{1}{x^2}y' - \frac{2}{x^3}y}_{} = \cos(x)$$

$$= \frac{d}{dx}\left(y \cdot \frac{1}{x^2}\right) = \cos(x) \Rightarrow \boxed{y \cdot \frac{1}{x^2} = \sin(x) + C}$$

(implicit)

or, solve for y:

$$\boxed{y = x^2\sin(x) + Cx^2} \text{ (explicit)}$$

Check: $y = x^2 \sin(x) + cx^2$

$\Rightarrow y' = 2x\sin(x) + x^2\cos(x) + 2cx$

Plug into $\frac{1}{x}\,dy = \left(x\cos(x) + \frac{2y}{x^2}\right)dx$

$$\frac{1}{x}\,y' = x\cos(x) + \frac{2y}{x^2}$$

$$\frac{1}{x}\left(2x\sin(x) + x^2\cos(x) + 2cx\right) \stackrel{?}{=} x\cos(x) + \frac{2\left(x^2\sin(x) + cx^2\right)}{x^2}$$

$$2\sin(x) + x\cos(x) + 2c \underset{?}{=} x\cos(x) + 2\sin(x) + 2c$$

yes