# Summary of Laplace Properties

## TRANSLATION THM. I.

$$\mathcal{L}\{e^{at}f(t)\} = \mathcal{L}\{f(t)\}\big|_{s\to s-a}$$

**Ex:** $\mathcal{L}\{e^{2t}\sin(4t)\}$

$= \mathcal{L}\{\sin(4t)\}\big|_{s\to s-2}$

$= \frac{4}{s^2+16}\Big|_{s\to s-2} = \frac{4}{(s-2)^2+16}$

! Frequency domain shifts too

$s>0 \mapsto s-2>0 \quad \boxed{s>2}$

(freq. domain of sin, used above)

$$\mathcal{L}^{-1}\{F(s)\big|_{s\to s-a}\} = e^{at}\mathcal{L}^{-1}\{F(s)\}$$

**Ex:** $\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^5}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{1}{s^5}\Big|_{s\to s-1}\right\}$

$= e^t\,\mathcal{L}^{-1}\left\{\frac{1}{s^5}\right\}$

$= e^t\,\frac{1}{4!}t^4$

## TRANSLATION THM. II

$$\mathcal{L}\{f(t-a)u_a(t)\} = e^{-as}\mathcal{L}\{f(t)\}$$

**Ex:** $f(t) = \begin{cases}(t-3)^2, & t\geq 2\\ 0, & 0\leq t<2\end{cases}$

$f(t) = (t-3)^2\,u_2(t)$

$= [(t-2)-1]^2\,u_2(t)$

$\mathcal{L}\{f(t)\} = e^{-2s}\mathcal{L}\{(t-1)^2\}$

$= e^{-2s}\mathcal{L}\{t^2-2t+1\}$

$= e^{-2s}\left(\frac{2}{s^3}-\frac{2}{s^2}+\frac{1}{s}\right) \quad (s>0)$

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = \mathcal{L}^{-1}\{F(s)\}\big|_{t\to t-a}\cdot u_a(t)$$

**Ex:** $\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2-10}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{1}{s^2-10}\right\}\Big|_{t\to t-2}\cdot u_2(t)$

$= \frac{1}{\sqrt{10}}\sinh(\sqrt{10}\,t)\Big|_{t\to t-2}u_2(t)$

$= \frac{1}{\sqrt{10}}\sinh(\sqrt{10}(t-2))\,u_2(t).$

## Derivatives of Laplace Transforms

$$\mathcal{L}\{t^n f(t)\} = (-1)^n\frac{d^n}{ds^n}\big[\underbrace{F(s)}_{\mathcal{L}\{f(t)\}}\big]$$

(use when there is an $t^n$ and no other options)

**Ex.** $\mathcal{L}\{t^2\cos(3t)\} = (-1)^2\frac{d^2}{ds^2}\mathcal{L}\{\cos(3t)\}$

$= \frac{d}{ds^2}\left(\frac{s}{s^2+9}\right) = \frac{d}{ds}\left(\frac{s^2+9-s\cdot 2s}{(s^2+9)^2}\right)$

$= \frac{d}{ds}\left(\frac{-s^2+9}{(s^2+9)^2}\right) = \frac{-2s(s^2+9)^2-(-s^2+9)\cdot 2(s^2+9)\cdot 2s}{(s^2+9)^4}$

$= \ldots$

## Laplace Transforms of Derivatives

$\mathcal{L}\{y'(t)\} = sY(s) - y(0)$

$\mathcal{L}\{y''(t)\} = s^2Y(s) - sy(0) - y'(0)$

$\vdots$

Use when solving ODEs w/ Laplace

## Laplace & Periodic Functions

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)\,dt$$

$$= \frac{1}{1-e^{-sT}}\mathcal{L}\left\{f(t)\big(1-u_T(t)\big)\right\}$$

Use when taking Laplace of periodic function $f$ w/ period $T$

Advantage: only have to integrate over 1 period.

## Convolution:

$$(f*g)(t) := \int_0^t f(t-y)g(y)\,dy = (g*f)(t)$$

### $\mathcal{L}\{f*g\} = \mathcal{L}\{f\}\mathcal{L}\{g\}$

Use when solving an equation where a convolution is involved (so usually w/ an integral), like RLC circuit equations or Volterra integral equations.

**Ex:** Solve for $x(t)$:

$$x(t) = \sin(2t) + \underbrace{\int_0^t \cos(2t-2y)x(y)\,dy}_{\cos(2t)*x(t)}$$

Take Laplace of everything:

$$X(s) = \frac{2}{s^2+4} + \underbrace{\mathcal{L}\{\cos(2t)*x(t)\}}_{=\frac{s}{s^2+4}\cdot X(s)}$$

$$\left(1-\frac{s}{s^2+4}\right)X(s) = \frac{2}{s^2+4}$$

$$X(s) = \frac{2}{\cancel{s^2+4}} \cdot \frac{\cancel{s^2+4}}{s^2+4-s}$$

$$X(s) = \frac{2}{s^2-s+4}$$

$$s^2-s+4 = \left(s^2-s+\frac{1}{4}\right)+4-\frac{1}{4}$$

$$= (s-1/2)^2 + \frac{15}{4}$$

$$x(t) = 2\mathcal{L}^{-1}\left\{\frac{1}{(s-1/2)^2+\frac{15}{4}}\right\}$$

$$= 2\mathcal{L}^{-1}\left\{\frac{1}{s^2+15/4}\Big|_{s\to s-1/2}\right\}$$

$$= 2e^{1/2t}\frac{1}{\frac{2}{\sqrt{15}}}\sin\left(\frac{\sqrt{15}}{2}t\right)$$

### $\mathcal{L}^{-1}\{F(s)G(s)\} = f(t)*g(t)$

Use when:

- Finding $\mathcal{L}^{-1}$ of a product & there are no easier options

**Ex:** $\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\} = \mathcal{L}^{-1}\{F(s)G(s)\}$,

where $F(s) = G(s) = \frac{1}{s^2+1} \Rightarrow$

$$f(t) = g(t) = \sin(t)$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\} = \sin(t)*\sin(t)$$

$\vdots$

compute...

- Solving an ODE via Laplace, with unknown forcing function $g(t)$:

**Ex:** $y'(t)+y(t) = g(t)$ ; $y(0)=1$

$$sY(s)-y(0)+Y(s) = G(s)$$

$$(s+1)Y(s) = G(s)+1$$

$$Y(s) = \frac{G(s)+1}{s+1}$$

$$\Rightarrow y(t) = \underbrace{\mathcal{L}^{-1}\left\{\frac{G(s)}{s+1}\right\}}_{\mathcal{L}^{-1}\{F(s)G(s)\} = (f*g)(t)} + \underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}}_{e^{-t}}$$

where $F(s) = \frac{1}{s+1} \Rightarrow f(t) = e^{-t}$

$$\Rightarrow y(t) = e^{-t} + (f*g)(t)$$

where $f(t) = e^{-t}$