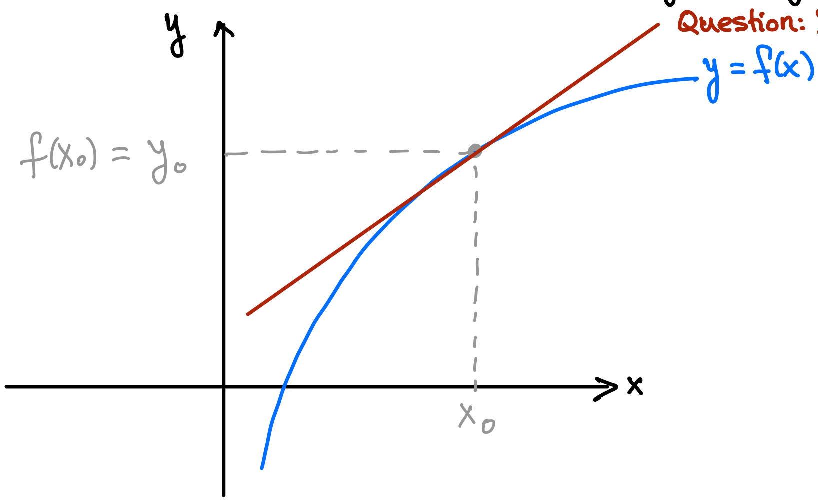
§2.6. Exact Equations

~> Multivariable Calculus Reminders ~

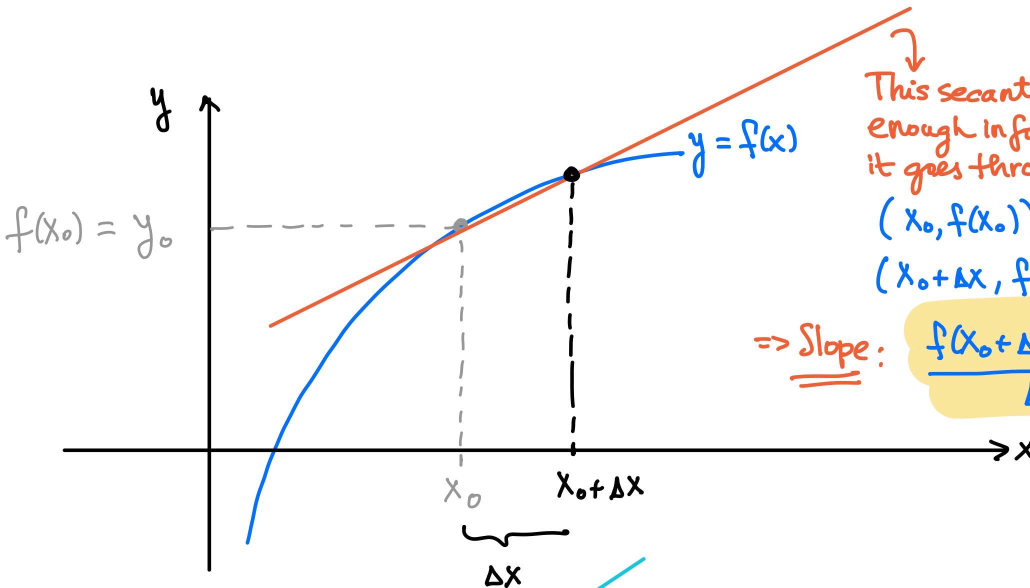
Recall the derivative in Calculus 1: $y=f(x)$ gets plotted in the plane:

Question: Slope of this tangent line?



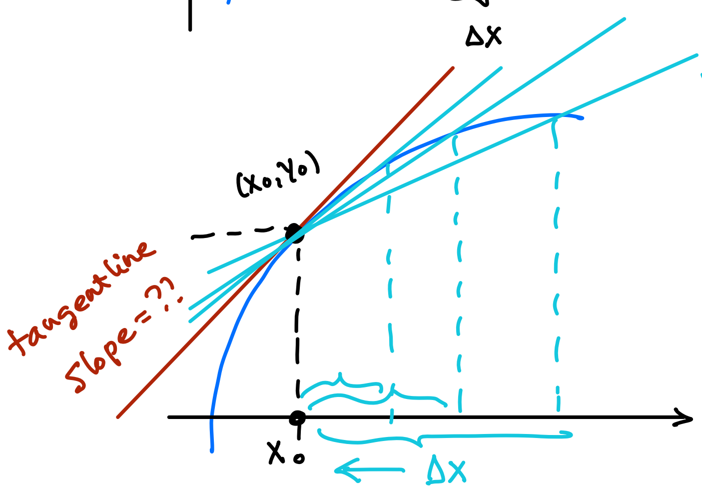
Problem: All we know is that the line passes through $(x_0, y_0=f(x_0))$. That's not enough information to determine the line!

Solution: Look at secant lines instead!



This secant line has enough information: it goes through the points: $(x_0, f(x_0))$ & $(x_0+\Delta x, f(x_0+\Delta x))$

=> Slope: $\dfrac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$

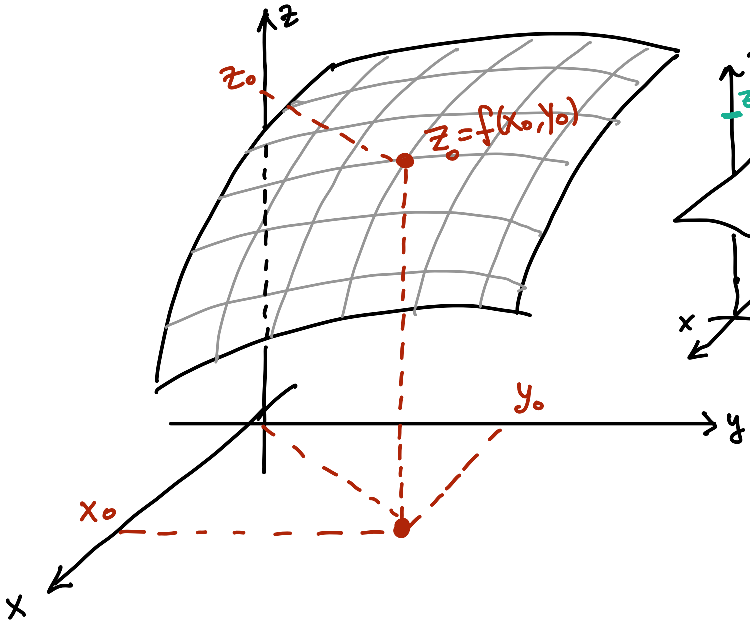


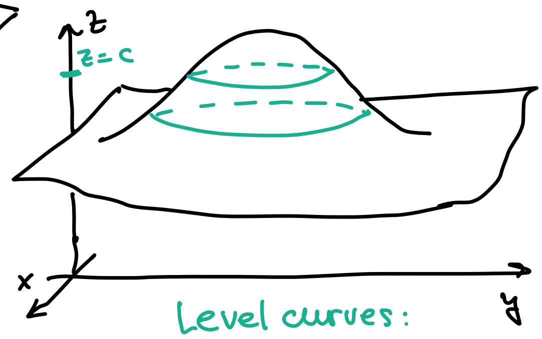
Secant lines collapse onto the tangent line as $\Delta x \to 0$!

=> Slope of the tangent line:

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$$

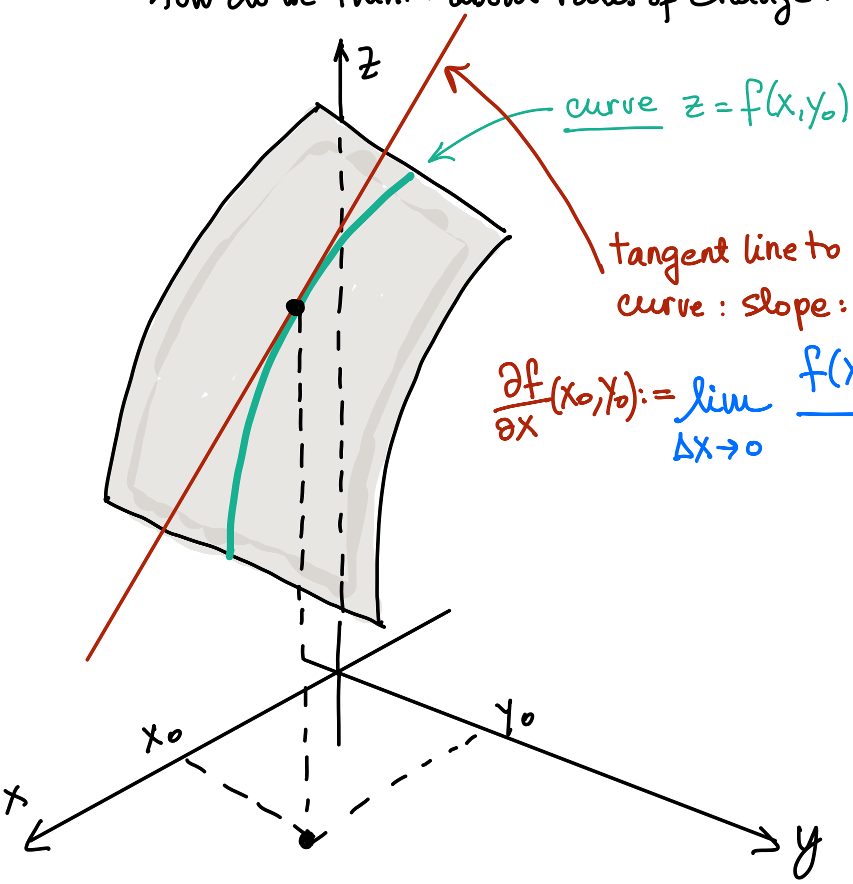
Now move to functions $z = f(x,y)$ of <u>two variables</u>, which get plotted in the (xyz) plane => graphs are now <u>surfaces</u> not curves!





Level curves:
$z = c$
or $f(x,y) = c$
(section by horizontal plane)

How do we think about rates of change?



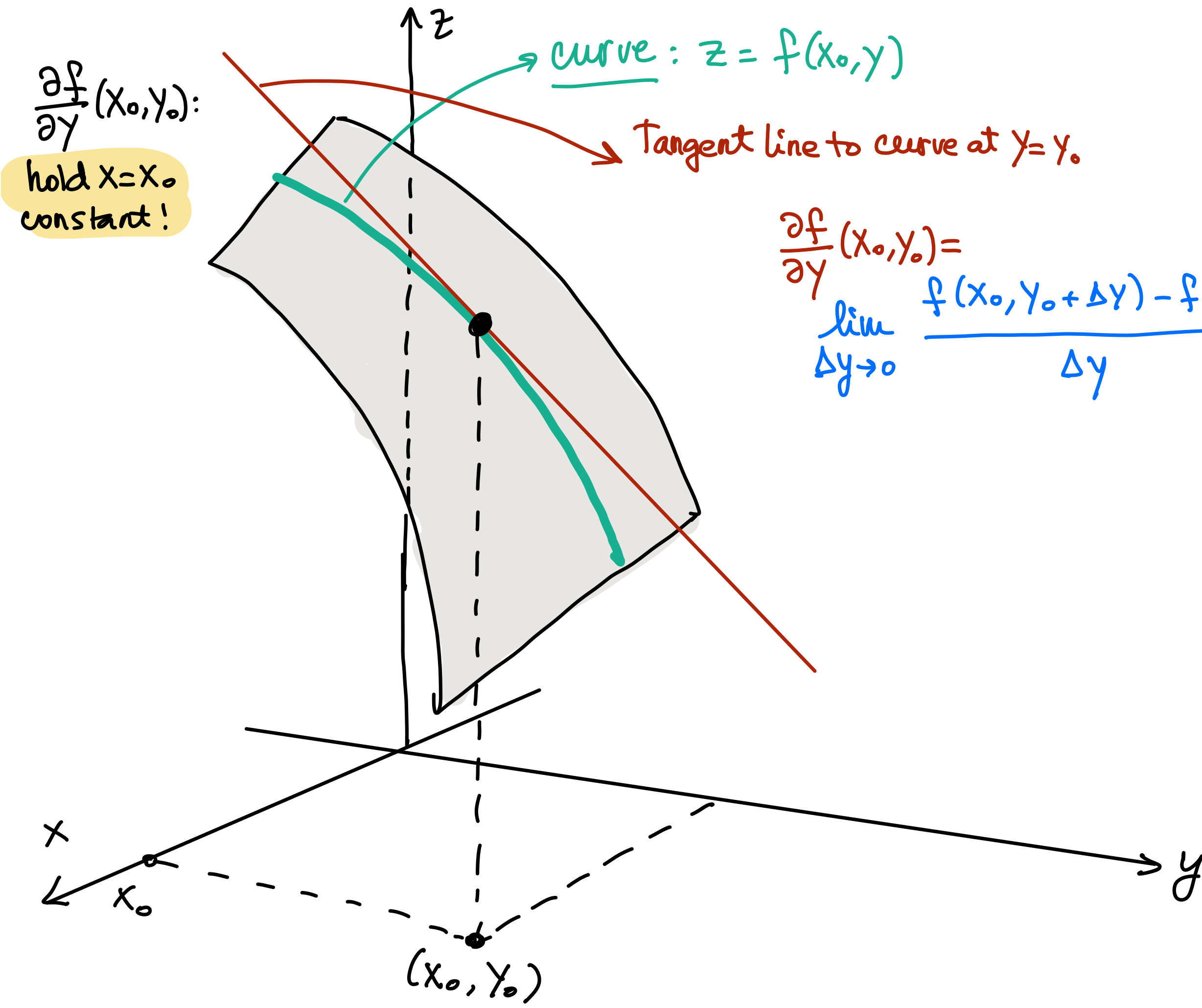
tangent line to curve : slope:

$$\frac{\partial f}{\partial x}(x_0,y_0) := \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x, y_0) - f(x_0,y_0)}{\Delta x}$$

Partial derivative

$$\frac{\partial f}{\partial x}(x_0, y_0)$$

Hold $y = y_0$ constant!



$$\frac{\partial f}{\partial y}(x_0, y_0) = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}$$

To compute $\frac{\partial f}{\partial x}$: hold all other variables (y, z etc.) constant & differentiate dx :

$$f(x,y) = x^2 y + \sin(y)$$

$$\frac{\partial f}{\partial x} = 2xy$$

$$\frac{\partial f}{\partial y} = x^2 + \cos(y)$$

$$f(x,y,z) = xy^2 z + e^{xz}$$

$$\frac{\partial f}{\partial x} = y^2 z + z e^{xz}$$

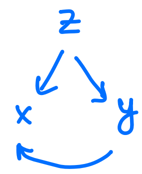
$$\frac{\partial f}{\partial y} = 2xyz$$

$$\frac{\partial f}{\partial z} = xy^2 + xe^{xz}$$

Multivariable Chain Rule

$z = f(x,y)$; y also depends on x

$z = f(x, y(x))$



$$\boxed{\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}}$$

Example: $f(x,y) = 2x + y^2$

$y = x^3 + 1$

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}$$

$$= (2) + (2y \cdot 3x^2)$$

$$= 2 + 2(x^3+1)\,3x^2$$

$\Rightarrow f(x,y) = 2x + (x^3+1)^2 = f(x)$

$$\frac{df}{dx} = 2 + 2(x^3+1) \cdot 3x^2$$

Same thing!

How do we need this?

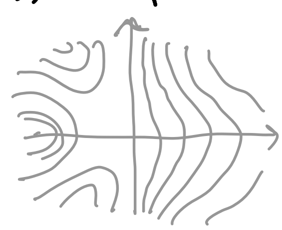
Ex]: $2x + y^2 + 2xy \cdot y' = 0$ $\rightarrow$ Not linear (in x or y), not separable

Say we have a function:

$f(x,y) = x^2 + xy^2$ $\Rightarrow$ The ODE becomes

$$\frac{\partial f}{\partial x} = 2x + y^2$$

$$\frac{\partial f}{\partial y} = 2xy$$



$$\underbrace{\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx}}_{\text{Chain Rule: } \frac{df}{dx}} = 0$$

Chain Rule: $\frac{df}{dx} = 0$

$\Rightarrow f(x,y) = \text{constant}$!

$\Rightarrow$ Solution: (implicit) $\boxed{x^2 + xy^2 = c}$ (just the level curves of f)

Exact Equations: when & how we can do this procedure.