Some Linear Algebra:

2×2 Determinants: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

Cramer's Rule: Say we have a system of algebraic equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases} \qquad A := \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$A_1 := \begin{pmatrix} \boxed{\begin{matrix} b_1 \\ b_2 \end{matrix}} & \begin{matrix} a_{12} \\ a_{22} \end{matrix} \end{pmatrix}; \quad A_2 := \begin{pmatrix} \begin{matrix} a_{11} \\ a_{21} \end{matrix} & \boxed{\begin{matrix} b_1 \\ b_2 \end{matrix}} \end{pmatrix}$$

$$\boxed{\begin{aligned} x_1 &= \frac{\det(A_1)}{\det(A)} \\ x_2 &= \frac{\det(A_2)}{\det(A)} \end{aligned}}$$

$$\begin{cases} x_1 + 2x_2 = 3 \\ 2x_1 - x_2 = 1 \end{cases}$$

$$\det(A) = \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} = -5$$

$$\det(A_1) = \begin{vmatrix} 3 & 2 \\ 1 & -1 \end{vmatrix} = -5 \qquad x_1 = x_2 = 1$$

$$\det(A_2) = \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = -5$$

Generally:
$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \qquad\qquad\qquad\qquad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n \end{cases}$$

$$\det(A) := \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

determinant of matrix of coefficients

For every $1 \le k \le n$: $\det(A_k) := \begin{vmatrix} a_{11} & \dots & \boxed{b_1} & \dots & a_{1n} \\ a_{21} & \dots & \boxed{b_2} & \dots & a_{2n} \\ \vdots & & \vdots & & \\ a_{n1} & \dots & \boxed{b_n} & \dots & a_{nn} \end{vmatrix}$

replace k^{th} column of A by column $\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$

Cramer's Rule: If $\det(A) \neq 0$, unique solution:
$$\begin{cases} x_1 = \dfrac{\det(A_1)}{\det(A)} \\ \vdots \\ x_k = \dfrac{\det(A_k)}{\det(A)} \\ \vdots \\ x_n = \dfrac{\det(A_n)}{\det(A)} \end{cases}$$

Homogeneous systems

$$\begin{cases} a_{11}x_1 + \ldots + a_{1n}x_n = 0 \\ a_{21}x_1 + \ldots + a_{2n}x_n = 0 \\ \vdots \qquad\qquad\quad \vdots \\ a_{n1}x_1 + \ldots + a_{nn}x_n = 0 \end{cases}$$

↓

always possess the trivial solution $x_1 = x_2 = \ldots = x_n = 0$.

→ If $\det(A) \neq 0$, by Cramer's rule, the trivial solution is the only solution.

If $\boxed{\det(A) = 0}$

- homogeneous systems : ∞-many solutions
- non-homogeneous systems : can have ∞-many, or no solutions.

3x3 determinants: (Reminder)

$$\begin{vmatrix} 2 & 4 & 7 \\ 1 & 2 & 3 \\ 1 & 5 & 3 \end{vmatrix}$$

Expand on 1st row:

$$\begin{array}{c} \begin{array}{ccc} \oplus & \ominus & \oplus \end{array} \\ \begin{vmatrix} 2 & 4 & 7 \\ 1 & 2 & 3 \\ 1 & 5 & 3 \end{vmatrix} \end{array} = +2 \cdot \begin{vmatrix} 2 & 3 \\ 5 & 3 \end{vmatrix} - 4 \cdot \begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} + 7 \cdot \begin{vmatrix} 1 & 2 \\ 1 & 5 \end{vmatrix}$$

$$= 2 \cdot (-9) - 4 \cdot 0 + 7 \cdot 3 = \boxed{3}$$

or, you can expand on any row & column

Expand on 2nd column:

$$\begin{vmatrix} 2 & \ominus\, 4 & 7 \\ 1 & \oplus\, 2 & 3 \\ 1 & \ominus\, 5 & 3 \end{vmatrix} = -4 \begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} + 2 \begin{vmatrix} 2 & 7 \\ 1 & 3 \end{vmatrix} - 5 \begin{vmatrix} 2 & 7 \\ 1 & 3 \end{vmatrix}$$

$$= -4 \cdot 0 + 2 \cdot (-1) - 5 \cdot (-1) = \boxed{3}$$

* Useful when you have 0's:

for example, for $\begin{array}{cc} \begin{vmatrix} 1 & 2 & 0 \\ 3 & 1 & 0 \\ -1 & 4 & 2 \end{vmatrix} & \begin{array}{c} \oplus \\ \ominus \\ \oplus \end{array} \end{array}$ it's fastest to expand on the 3rd column:

$$+2 \cdot \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = -10$$

Linearly Dependent Functions (general def.)

A set of functions $f_1(x), f_2(x), \dots, f_n(x)$ is said to be **linearly dependent** on an interval I if there exist constants $c_1, c_2, \dots, c_n$ NOT ALL ZERO such that:

$$c_1 f_1(x) + \dots + c_n f_n(x) = 0 \quad \text{for all } x \in I.$$

Otherwise, they are called **linearly independent.**

→ i.e.: $f_1, \dots, f_n$ are linearly independent on an interval I if the only constants $c_1, \dots, c_n$ for which

$$c_1 f_1(x) + \dots + c_n f_n(x) = 0, \text{ for all } x \in I$$

are $c_1 = c_2 = \dots = c_n = 0$.

→ Linear dependence means at least one function is a linear combination of the others

Ex.:
$$\left.\begin{aligned} f_1(x) &= \sqrt{x} + 5 \\ f_2(x) &= \sqrt{x} + 5x \\ f_3(x) &= x - 1 \\ f_4(x) &= x^2 \end{aligned}\right\}$$
linearly dependent on $(0, \infty)$ b/c f_2 can be written as a linear combination of f_1, f_3, f_4:

$$f_2(x) = 1 \cdot f_1(x) + 5 \cdot f_3(x) + 0 \cdot f_4(x) \text{ holds for } \underline{\text{all}}\ x \in (0, \infty).$$

THE WRONSKIAN: If n functions $f_1, f_2, \dots, f_n$ have at least $(n-1)$ derivatives, their Wronskian is the $n \times n$ determinant:

$$\begin{vmatrix} f_1 & f_2 & \dots & f_n \\ f_1' & f_2' & \dots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \dots & f_n^{(n-1)} \end{vmatrix} =: W(f_1, \dots, f_n)(x)$$

THEOREM: If the Wronskian $W(f_1, \ldots, f_n)$ of n functions $f_1, \ldots, f_n$ is not zero for at least one point in an interval I, then the set of functions $\{f_1, f_2, \ldots, f_n\}$ is linearly independent on the interval I.

Proof for case $n=2$: By contradiction.

Assume that $W(f_1(x), f_2(x)) \neq 0$ for a fixed point $x_0 \in I$:

$$W(f_1(x_0), f_2(x_0)) \neq 0$$

and that f_1, f_2 are linearly dependent on I.

$\Rightarrow$ there exist constants c_1, c_2 not both zero such that

$$c_1 f_1(x) + c_2 f_2(x) = 0, \text{ for all } x \in I$$

$$\Rightarrow \quad c_1 f_1'(x) + c_2 f_2'(x) = 0 \quad \text{also, for all } x \in I$$

$\Rightarrow$ system of linear equations: (w/ unknowns c_1, c_2)

$$\begin{cases} c_1 f_1(x_0) + c_2 f_2(x_0) = 0 \\ c_1 f_1'(x_0) + c_2 f_2'(x_0) = 0 \end{cases}$$

has a nontrivial solution $\Rightarrow \begin{vmatrix} f_1(x_0) & f_2(x_0) \\ f_1'(x_0) & f_2'(x_0) \end{vmatrix} = 0 \Rightarrow$ Contradiction! ∎

Examples:

① $f_1(x)=e^{2x}$; $f_2(x)=e^{-x}$

$$W(f_1,f_2)=\begin{vmatrix} e^{2x} & e^{-x} \\ 2e^{2x} & -e^{-x} \end{vmatrix} = -e^{x}-2e^{x}=-3e^{x}\neq 0 \text{ for all real } x$$

$\Rightarrow \{f_1,f_2\}$ lin. indp. on $\mathbb{R}$.

② $f_1(x)=\sin^2(x)$; $f_2(x)=\cos^2(x)$

$$W(f_1,f_2)=\begin{vmatrix} \sin^2(x) & \cos^2(x) \\ 2\sin(x)\cos(x) & -2\sin(x)\cos(x) \end{vmatrix} = \begin{vmatrix} \sin^2(x) & \cos^2(x) \\ \sin(2x) & -\sin(2x) \end{vmatrix}$$

$$= -\sin(2x)\cdot\sin^2(x)-\sin(2x)\cdot\cos^2(x)$$

$$= -\sin(2x)\underbrace{\left(\sin^2(x)+\cos^2(x)\right)}_{1}$$

$$= -\sin(2x)$$

$\hookrightarrow$ $\sin(2x)$ is non-zero at at least one point of $\mathbb{R}$ (∞-many actually)

$\Rightarrow$ linearly independent on $\mathbb{R}$.

③ $f_1(x)=\sin^2(x)$; $f_2(x)=\cos^2(x)$; $f_3(x)=2$

$$W(f_1,f_2,f_3)=\begin{vmatrix} \sin^2(x) & \cos^2(x) & 2 \\ \sin(2x) & -\sin(2x) & 0 \\ 2\cos(2x) & -2\cos(2x) & 0 \end{vmatrix}$$

$$= 2\begin{vmatrix} \sin(2x) & -\sin(2x) \\ 2\cos(2x) & -2\cos(2x) \end{vmatrix} = 2\big(-2\sin(2x)\cos(2x) + 2\sin(2x)\cos(2x)\big) = 0$$

$\Rightarrow$ Linearly dependent on $\mathbb{R}$ ($W=0$ for all real x)