Theorem 1: Consider n functions $f_1, f_2, \ldots, f_n$ on an interval I, where all functions f_i are continuous and n-times differentiable. If there exists a point $x_0 \in I$ such that $W(f_1,\ldots,f_n)(x_0) \neq 0$, then the set of functions $\{f_1,\ldots,f_n\}$ is <u>linearly independent on I</u>.

Proof for case $n=2$ (a proof by contradiction):

Suppose $x_0 \in I$ is such that $W(f_1,f_2)(x_0) \neq 0$ and f_1, f_2 are linearly dependent on I.
=> there exist constants C_1, C_2 <u>not both 0</u> such that:

$$C_1 f_1(x) + C_2 f_2(x) = 0, \text{ for all } x \in I$$
$$\Rightarrow \quad C_1 f_1'(x) + C_2 f_2'(x) = 0 \text{ for all } x \in I \text{ as well}$$

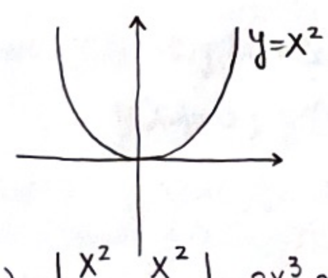
=> System of linear equations (w/ unknowns C_1, C_2): $\begin{cases} C_1 f_1(x_0) + C_2 f_2(x_0) = 0 \\ C_1 f_1'(x_0) + C_2 f_2'(x_0) = 0 \end{cases}$
has a <u>non-trivial solution</u> (C_1, C_2 not both 0)

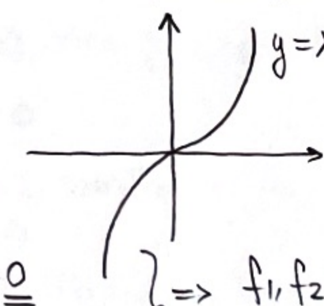
=> by Cramer's Rule: $\begin{vmatrix} f_1(x_0) & f_2(x_0) \\ f_1'(x_0) & f_2'(x_0) \end{vmatrix} = 0 \Rightarrow W(f_1,f_2)(x_0) = 0 \Rightarrow$ <u>Contradiction!</u> ■

Remark 1: The implication $[W(f_1,f_2)$ not identically 0 on $I] \Rightarrow [f_1, f_2$ lin. indp. on $I]$ generally does not go both ways, i.e.

A set of functions $\{f_1,\ldots,f_n\}$ can be linearly independent on an interval, and have a vanishing Wronskian throughout I.

<u>Example</u>: $f_1(x) = x^2$ and $f_2(x) = x|x|$ are obviously <u>linearly independent on $\mathbb{R}$</u>:





$y = x|x| = \begin{cases} x^2, & x \geq 0 \\ -x^2, & x < 0 \end{cases}$

For $x \geq 0$: $W(f_1,f_2) = \begin{vmatrix} x^2 & x^2 \\ 2x & 2x \end{vmatrix} = 2x^3 - 2x^3 = \underline{\underline{0}}$

For $x < 0$: $W(f_1,f_2) = \begin{vmatrix} x^2 & -x^2 \\ 2x & -2x \end{vmatrix} = -2x^3 + 2x^3 = \underline{\underline{0}}$

=> f_1, f_2 are linearly independent on $\mathbb{R}$, and yet $W(f_1,f_2)(x) = 0$ for all $x \in \mathbb{R}$.

Remark 2: Later on, in the method of "Variation of Parameters" for linear ODE's, we will end up using Cramer's Rule & dividing by $W(y_1,y_2)(x)$ on some interval I, where $\{y_1, y_2\}$ are a fundamental set of solutions to a homogeneous linear ODE. The theorem above <u>doesn't guarantee</u> we can safely do this for all $x \in I$, if all we know is that y_1 & y_2 are linearly independent. But it turns out that if we know further that y_1 & y_2 are solutions to a linear ODE, we can say much more:

Theorem 2: Suppose $y_1, y_2, \dots, y_n$ are solutions to an n^{th} order linear homogeneous ODE:

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

on an interval I. Then:

$$\boxed{\{y_1, y_2, \dots, y_n\} \text{ are linearly independent on } I} \iff \boxed{\begin{array}{c} W(y_1, y_2, \dots, y_n)(x) \neq 0 \\ \text{for all } x \in I \end{array}}$$

Proof for $n=2$:

($\Leftarrow$) • One implication is trivial: if $W(y_1,y_2)(x) \neq 0$ for all $x \in I$, then the assumptions of Theorem 1 are met $\Rightarrow$ $\{y_1, y_2\}$ are linearly independent.

($\Rightarrow$) • Show: if y_1, y_2 are linearly independent solutions to $\boxed{a_2(x)y'' + a_1(x)y' + a_0(x)y = 0}$ $(*)$ on I, then $W(y_1,y_2)(x) \neq 0$ for all $x \in I$.

Suppose not, i.e. suppose there is $x_0 \in I$ such that $W(y_1,y_2)(x_0) = 0$:

$$\begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} = 0$$

$\Rightarrow$ by Cramer's Rule, the homogeneous linear system (w/ unknowns c_1, c_2): $\begin{cases} c_1 y_1(x_0) + c_2 y_2(x_0) = 0 \\ c_1 y_1'(x_0) + c_2 y_2'(x_0) = 0 \end{cases}$

has a non-trivial solution, i.e. there exist c_1, c_2 not both 0 such that

$$\begin{cases} c_1 y_1(x_0) + c_2 y_2(x_0) = 0 \\ c_1 y_1'(x_0) + c_2 y_2'(x_0) = 0 \end{cases}$$

$\Rightarrow$ if we define a function $y(x) := c_1 y_1(x) + c_2 y_2(x)$, then y satisfies the initial conditions:

$$y(x_0) = 0\ ;\ y'(x_0) = 0$$

BUT the identically 0 function $y(x) = 0$, for all $x \in I$, satisfies $(*)$: $a_2 y'' + a_1 y' + a_0 y = 0$ and the initial conditions $y(x_0) = y'(x_0) = 0$

$\Rightarrow$ by the Existence & Uniqueness Theorem, $y \equiv 0$ is the only solution to this IVP

$$\left.\begin{array}{l} \Rightarrow c_1 y_1(x) + c_2 y_2(x) = 0 \text{ for all } x \in I \\ \qquad c_1, c_2 \text{ not both } 0 \end{array}\right\} \Rightarrow y_1, y_2 \text{ are linearly dependent on } I$$

$$\Downarrow$$

Contradiction!

■

Variation of Parameters

Setup: Linear second-order ODE:

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$$

put into standard form:

$$y'' + P(x)y' + Q(x)y = f(x) \qquad (*)$$

by dividing out $a_2(x)$, on an interval I where $a_2(x) \neq 0$. Assume $P(x), Q(x), f(x)$ are continuous on I.

→ Recall: under these assumptions, the general solution to $(*)$ is of the form

$$y = y_c + y_p$$

where y_c = complementary solution ; y_p = any particular solution.

→ When $P(x), Q(x)$ are constant, we know exactly how to find y_c.

→ Suppose $\{y_1, y_2\}$ is a fundamental set of solutions to the homogeneous form of $(*)$, i.e.

$$\begin{cases} y_1'' + P(x)y_1' + Q(x)y_1 = 0 \\ y_2'' + P(x)y_2' + Q(x)y_2 = 0 \end{cases}$$ and y_1, y_2 are linearly independent on I.

⇒ Complementary solution: $y_c = c_1 y_1 + c_2 y_2$

→ Q: Can we find two functions $u_1(x), u_2(x)$ so that

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$$

is a particular solution to $(*)$?

(looks just like y_c! Except the constants c_1, c_2 now vary, and are replaced by functions u_1, u_2)

→ Looking for 2 functions ⇒ will probably need 2 equations;

⤳ The obvious one: y_p must satisfy $(*)$.

⤳ The second one: an assumption we make to simplify the derivatives of y_p.

$$\boxed{y_p = u_1 y_1 + u_2 y_2} \Rightarrow y_p' = u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2'$$

$$= (u_1 y_1' + u_2 y_2') + \underbrace{(y_1 u_1' + y_2 u_2')}_{}$$

Assume this is 0: $\boxed{y_1 u_1' + y_2 u_2' = 0}$

$$\Rightarrow \boxed{y_p' = u_1 y_1' + u_2 y_2'}$$

$$\Rightarrow \boxed{y_p'' = u_1 y_1'' + u_1' y_1' + u_2' y_2' + u_2 y_2''}$$

Replace all in $(*)$

$$f(x) = y_p'' + Py_p' + Qy_p = \overbrace{u_1 y_1'' + u_1' y_1'} + \overbrace{u_2' y_2' + u_2 y_2''}$$
$$+ P(u_1 y_1' + u_2 y_2')$$
$$+ Q(u_1 y_1 + u_2 y_2)$$

$$= u_1 \underbrace{[y_1'' + Py_1' + Qy_1]}_{0} + u_2 \underbrace{[y_2'' + Py_2' + Qy_2]}_{0} + u_1' y_1' + u_2' y_2'$$

b/c y_1, y_2 are solutions to the homogeneous equation

=> Second equation: $y_1' u_1' + y_2' u_2' = f$

=> Linear system of equations to find u_1', u_2':

$$\begin{cases} y_1 u_1' + y_2 u_2' = 0 \\ y_1' u_1' + y_2' u_2' = f(x) \end{cases} \qquad \begin{cases} y_1(x) u_1'(x) + y_2(x) u_2'(x) = 0 \\ y_1'(x) u_1'(x) + y_2'(x) u_2'(x) = f(x) \end{cases}$$

For every $x \in I$, this is an <u>algebraic</u> linear system, w/ <u>unknowns</u> $u_1'(x), u_2'(x)$

=> <u>Cramer's Rule</u> $\quad u_1'(x) = \dfrac{W_1(x)}{W(x)}$; $u_2'(x) = \dfrac{W_2(x)}{W(x)}$

where

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$$ (the Wronskian of y_1, y_2)

and

$$W_1(x) = \begin{vmatrix} 0 & y_2(x) \\ f(x) & y_2'(x) \end{vmatrix} ; \quad W_2(x) = \begin{vmatrix} y_1(x) & 0 \\ y_1'(x) & f(x) \end{vmatrix}$$

→ Note: ok to divide by $W(x)$ here b/c $W(x) \neq 0$ for all $x \in I$, as the Wronskian of solutions to a <u>linear ODE</u>.

→ Also: Constants of integration are not needed when finding u_1 & u_2, because they would just duplicate y_c:

Say $u_1' = x$ => take $u_1 = \dfrac{x^2}{2}$.

Taking $u_1 = \dfrac{x^2}{2} + C$ would just lead to $u_1 y_1 = \dfrac{x^2}{2} y_1 + Cy_1$ already in y_c

Variation of Parameters: Summary

→ Used for: Linear ODEs (in "standard form"): $\boxed{y'' + P(x)y' + Q(x)y = f(x)}$

→ Steps:

① Find the complementary solution $y_c = c_1y_1 + c_2y_2$ (to the homogeneous ODE)

② Compute the Wronskian $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

③ Compute $\boxed{u_1' = \dfrac{W_1}{W}}$ & $\boxed{u_2' = \dfrac{W_2}{W}}$, where $W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}$, $W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$

④ Integrate u_1' & u_2' found above, to finally find the functions u_1, u_2

⟹ Particular Solution: $\boxed{y_p = u_1y_1 + u_2y_2}$

Example: Solve: $y'' - 4y' + 4y = (x+1)e^{2x}$ (already in standard form)

→ Complementary Solution:

Char. Eqn.: $m^2 - 4m + 4 = 0$
$(m-2)^2 = 0$
$m_1 = m_2 = 2$

$y_1 = e^{2x}$
$y_2 = xe^{2x}$

$y_c = c_1e^{2x} + c_2xe^{2x}$

→ Wronskian: $W(y_1, y_2) = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x} + 2xe^{2x} \end{vmatrix} = e^{4x}\begin{vmatrix} 1 & x \\ 2 & 1+2x \end{vmatrix} = e^{4x}(1+2x-2x) = e^{4x}$

$W(y_1, y_2) = e^{4x}$

→ $f(x) = (x+1)e^{2x}$

$W_1 = \begin{vmatrix} 0 & xe^{2x} \\ (x+1)e^{2x} & e^{2x} + 2xe^{2x} \end{vmatrix}$
$= e^{4x}\begin{vmatrix} 0 & x \\ x+1 & 1+2x \end{vmatrix}$
$= -x(x+1)e^{4x}$

⇓

$u_1' = -x(x+1)$
$= -x^2 - x$

⇓

$\boxed{u_1 = -\dfrac{x^3}{3} - \dfrac{x^2}{2}}$

$W_2 = \begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & (x+1)e^{2x} \end{vmatrix} = (x+1)e^{4x}$

⇓

$u_2' = x+1$

⇓

$\boxed{u_2 = \dfrac{x^2}{2} + x}$

⟹ Particular solution: $y_p = \left(-\dfrac{x^3}{3} - \dfrac{x^2}{2}\right)e^{2x} + \left(\dfrac{x^2}{2} + x\right)xe^{2x}$
$= \left(-\dfrac{x^3}{3} - \dfrac{x^2}{2} + \dfrac{x^3}{2} + x^2\right)e^{2x}$
$= \left(\dfrac{x^3}{6} + \dfrac{x^2}{2}\right)e^{2x}$

$y_p = \left(\dfrac{x^3}{6} + \dfrac{x^2}{2}\right)e^{2x}$

⟹ **Solution**: $y = c_1e^{2x} + c_2xe^{2x} + \left(\dfrac{x^3}{6} + \dfrac{x^2}{2}\right)e^{2x}$