# CHAPTER 6: The Laplace Transform

**Functions:** # number $\xrightarrow{\text{rule}}$ # number

$f(x) = x - 2$

input: $x = \underline{\underline{2}} \Rightarrow$ output $f(2) = \underline{\underline{0}}$

**Integral Transforms:** function $\xrightarrow{\int}$ function

$f(t) \longmapsto \int K(s,t) f(t)\, dt$

function of $t$ (time domain) [seconds] $\longmapsto$ function of $s$ (frequency domain) [1/second]

$K(s,t) =$ "Kernel"

**The Laplace Transform:**

$$\boxed{\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt}$$

Kernel: $K(s,t) = e^{-st}$

Notation: Lowercase letter $f(t)$ for the function being transformed, capital letter $F(s)$ for the resulting function (Laplace transform).

$$\mathcal{L}\{f(t)\} = F(s)\ ; \quad \mathcal{L}\{g(t)\} = G(s); \quad \mathcal{L}\{y(t)\} = Y(s) \ldots$$

**Example 1:** $\mathcal{L}\{1\}$ [this means Laplace transform of the constant function $f \equiv 1$, i.e. $f(t) = 1$ for all $t$]

$f(t) = 1$

$$F(s) = \mathcal{L}\{1\} = \int_0^\infty e^{-st} \cdot 1\, dt = \frac{-1}{s} e^{-st}\Big|_{t=0}^{t=\infty} = 0 - \frac{-1}{s} \cdot 1 = \frac{1}{s}$$

only if $s > 0$!

$$\boxed{\mathcal{L}\{1\} = \frac{1}{s}; \quad s > 0}$$

Recall improper integrals:

$\int_0^\infty e^{-st}\,dt$ means $\lim_{u\to\infty}\int_0^u e^{-st}\,dt = \lim_{u\to\infty}\left(\frac{-1}{s}e^{-st}\Big|_{t=0}^{t=u}\right)$

$$= \lim_{u\to\infty}\left(\frac{-1}{s}e^{-su} - \frac{-1}{s}e^{-s\cdot 0}\right)$$

$$= \underbrace{\lim_{u\to\infty}\left(\frac{-1}{s}e^{-su}\right)}_{\text{This limit is 0 if } s>0} + \frac{1}{s}$$

$\boxed{e^{-\infty}=0}$ $\boxed{e^{\infty}=\infty}$

This limit is 0 if $s>0$

Example 2: $\mathcal{L}\{t\}$

$F(s) = \mathcal{L}\{t\} = \int_0^\infty e^{-st}\cdot t\,dt$

$f(t)=t$

$$= \left(\frac{-t}{s}e^{-st} - \frac{1}{s^2}e^{-st}\right)\Big|_{t=0}^{\infty}$$

$$= (0-0) - \left(0 - \frac{1}{s^2}\right) = \frac{1}{s^2}$$

$\boxed{\mathcal{L}\{t\} = \frac{1}{s^2}}$ $s>0$

$\int e^{-st}\cdot t\,dt$

$u=t \quad dv = e^{-st}$

$du = dt \quad v = \frac{-1}{s}e^{-st}$

$= \frac{-t}{s}e^{-st} - \int \frac{-1}{s}e^{-st}\,dt$

$= \frac{-t}{s}e^{-st} - \frac{1}{s^2}e^{-st}$

$$\lim_{u\to\infty}\left(\frac{-u}{s}e^{-su} - \frac{1}{s^2}e^{-su}\right)$$

$\frac{u}{e^{su}} \xrightarrow[u\to\infty]{} 0$ $\quad e^{-\infty}=0$

$\frac{\text{polynomial}}{\text{exponential}}$

formally: $\lim_{u\to\infty}\frac{u}{e^{su}} \underset{\frac{\infty}{\infty}}{\overset{\text{l'H}}{=}} \lim_{u\to\infty}\frac{1}{se^{su}} = 0 \quad \left(\frac{1}{\infty}\right)$

Example 3: $\mathcal{L}\{e^{-3t}\}$ $s>-3$ (otherwise it diverges)

$$F(s)=\mathcal{L}\{e^{-3t}\}=\int_0^\infty e^{-st}\cdot e^{-3t}\,dt=\int_0^\infty e^{-(s+3)t}\,dt$$

$f(t)=e^{-3t}$

$$=\frac{-1}{s+3}e^{-(s+3)t}\Big|_{t=0}^{\infty}=0-\frac{-1}{s+3}=\frac{1}{s+3}$$

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## Linearity of the Laplace Transform

$$\boxed{\begin{aligned}\mathcal{L}\{\alpha f(t)+\beta g(t)\}&=\alpha\mathcal{L}\{f(t)\}+\beta\mathcal{L}\{g(t)\}\\&=\alpha F(s)+\beta G(s)\end{aligned}}$$

So, for example, we already know: $\mathcal{L}\{1\}=\frac{1}{s}$; $s>0$

$\mathcal{L}\{t\}=\frac{1}{s^2}$; $s>0$

$\mathcal{L}\{e^{-3t}\}=\frac{1}{s+3}$; $s>-3$

$\Rightarrow$ we can deduce from these, for example:

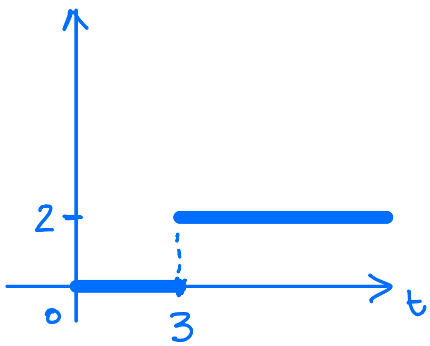
$$\mathcal{L}\{3\}=3\cdot\mathcal{L}\{1\}=\frac{3}{s};\ s>0$$

$$\mathcal{L}\{4+2t\}=4\mathcal{L}\{1\}+2\mathcal{L}\{t\}=\frac{4}{s}+\frac{2}{s^2};\ s>0$$

$$\mathcal{L}\{6e^{-3t}\}=6\mathcal{L}\{e^{-3t}\}=\frac{6}{s+3};\ s>-3$$

## Example 4: A step function:

$$f(t)=\begin{cases}0, & \text{if } 0\le t<3\\ 2, & \text{if } 3\le t<\infty\end{cases}$$




$$F(s)=\int_0^\infty e^{-st}\cdot f(t)\,dt$$

$$=\underbrace{\int_0^3 e^{-st}\cdot 0\,dt}_{0}+\int_3^\infty e^{-st}\cdot 2\,dt$$

$$=2\int_3^\infty e^{-st}\,dt$$

$$=2\,\frac{-1}{s}e^{-st}\Big|_{t=3}^{\infty}$$

$$=0-\frac{-2}{s}e^{-3s}\qquad (s>0)$$

$$=\frac{2}{s}e^{-3s}$$

$$\boxed{F(s)=\frac{2}{s}e^{-3s};\quad s>0}$$

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## Why do we care about $\mathcal{L}$ in ODE theory?

Integration by Parts:

$$\int fg'=fg-\int f'g$$

$$\mathcal{L}\{f'(t)\}=\int_0^\infty e^{-st}\cdot f'(t)\,dt$$

$$=e^{-st}\cdot f(t)\Big|_{t=0}^{\infty}-\int_0^\infty (e^{-st})'f(t)\,dt$$

$$\underset{(s>0)}{=}\big(0-f(0)\big)-\int_0^\infty -s\,e^{-st}f(t)\,dt$$

$$=-f(0)+s\underbrace{\int_0^\infty e^{-st}f(t)\,dt}_{\mathcal{L}\{f\}=F(s)}$$

$$\Rightarrow \boxed{\mathcal{L}\{f'(t)\} = sF(s) - f(0)}$$

So, take for example the ODE:

$$\frac{dy}{dt} + 3y = 2t \quad ; \; y(0) = 6$$

Take Laplace transform of the whole thing:

$$\underbrace{\mathcal{L}\{y'\}} + 3\underbrace{\mathcal{L}\{y\}} = 2\mathcal{L}\{t\}$$

recall this is really $y(t)$

$\mathcal{L}\{y(t)\} = Y(s)$

$$\left(sY(s) - \underbrace{y(0)}_{6}\right) + \left(3Y(s)\right) = \frac{2}{s^2}$$

$$(s+3)Y(s) - 6 = \frac{2}{s^2}$$

$$(s+3)Y(s) = 6 + \frac{2}{s^2} = \frac{6s^2+2}{s^2} \Rightarrow \boxed{Y(s) = \frac{6s^2+2}{s^2(s+3)}}$$

This led us to the conclusion:

The solution $y(t)$ to this IVP is the function $y(t)$ whose Laplace transform is $\frac{6s^2+2}{s^2(s+3)}$.

$\Rightarrow$ To find $y(t)$, we will take the inverse Laplace transform:

$$y(t) = \mathcal{L}^{-1}\left\{\frac{6s^2+2}{s^2(s+3)}\right\}.$$

Point: Taking the Laplace transform of a <u>differential equation</u> turns it into an <u>algebraic equation</u> (much easier to solve).

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We need to build a library of Laplace transforms. So far:

$$\mathcal{L}\{1\} = \frac{1}{s}\,;\ s>0.$$

$$\mathcal{L}\{t\} = \frac{1}{s^2}\,;\ s>0$$

$$\mathcal{L}\{t^2\} = \frac{2}{s^3}\,;\ s>0$$

$$\mathcal{L}\{t^2\} = \int_0^\infty e^{-st}\cdot t^2\,dt$$

$$= \underbrace{\frac{-t^2}{s}e^{-st}\Big|_{t=0}^{\infty}}_{0 \text{ if } s>0} + \frac{2}{s}\underbrace{\int_0^\infty t e^{-st}\,dt}_{\mathcal{L}\{t\}}$$

$$\int e^{-st}\cdot t^2\,dt$$

$$u=t^2;\ dv=e^{-st}$$

$$du=2t\,dt\,;\ v=-\frac{1}{s}e^{-st}$$

$$-\frac{t^2}{s}e^{-st}+\frac{2}{s}\int t e^{-st}\,dt$$

$$\Rightarrow \mathcal{L}\{t^2\} = \frac{2}{s}\mathcal{L}\{t\} = \frac{2}{s^3}$$

$$\boxed{\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}\ ; \quad n = 1, 2, 3, \ldots \quad (s>0)}$$

Proof (by induction):

Statement (n): $\mathcal{L}\{t^n\} = \dfrac{n!}{s^{n+1}}$ $\qquad n = 1, 2, 3, \ldots$

① The Statement is True for $n=1$ – proved earlier, and also for $n=2$.

② Induction:

> Suppose Statement (n) is true.
> Show that Statement (n+1) is then also true.

$$\underset{\text{True}}{n=1} \Rightarrow n=2 \Rightarrow n=3 \Rightarrow n=4 \Rightarrow \ldots\ldots$$

So: Suppose $\mathcal{L}\{t^n\} = \dfrac{n!}{s^{n+1}}$ is true. Then Look at $\mathcal{L}\{t^{n+1}\}$:

$$\mathcal{L}\{t^{n+1}\} = \int_0^\infty e^{-st} \cdot t^{n+1}\, dt$$

$$= \underbrace{\frac{-t^{n+1}}{s} e^{-st} \Big|_{t=0}^{\infty}}_{0 \text{ if } s>0} + \frac{n+1}{s} \underbrace{\int_0^\infty t^n e^{-st}\, dt}_{\mathcal{L}\{t^n\}}$$

$$\int e^{-st} \cdot t^{n+1}\, dt$$

$$u = t^{n+1}; \quad dv = e^{-st}$$

$$du = (n+1)t^n; \quad v = \frac{-1}{s} e^{-st}$$

$$= -\frac{t^{n+1}}{s} e^{-st} + \frac{n+1}{s} \int t^n e^{-st} dt$$

$$\Rightarrow \mathcal{L}\{t^{n+1}\} = \frac{n+1}{s} \cdot \underbrace{\mathcal{L}\{t^n\}}_{= \frac{n!}{s^{n+1}} \text{ by assumption}} = \frac{n+1}{s} \frac{n!}{s^{n+1}} = \frac{(n+1)!}{s^{n+2}}$$

$$\Rightarrow \mathcal{L}\{t^{n+1}\} = \frac{(n+1)!}{s^{(n+1)+1}} \qquad \text{Statement } (n+1) \ \checkmark$$

$\blacksquare$