Last time: Laplace Transform definition:

$$\boxed{\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt}$$

Some examples we saw:

$$\boxed{\mathcal{L}\{1\} = \frac{1}{s};\quad s>0}$$

$$\boxed{\mathcal{L}\{t\} = \frac{1}{s^2};\quad s>0}$$

$$\mathcal{L}\{t^2\} = \int_0^\infty e^{-st}\cdot t^2\,dt$$

$$= \underbrace{\frac{-t^2}{s}e^{-st}\Big|_{t=0}^{\infty}}_{0 \text{ if } s>0} + \frac{2}{s}\underbrace{\int_0^\infty t e^{-st}\,dt}_{\mathcal{L}\{t\}}$$

$$\int e^{-st}\cdot t^2\,dt$$

$$u = t^2;\ dv = e^{-st}$$

$$du = 2t\,dt;\ v = \frac{-1}{s}e^{-st}$$

$$\frac{-t^2}{s}e^{-st} + \frac{2}{s}\int t e^{-st}\,dt$$

$$\Rightarrow \mathcal{L}\{t^2\} = \frac{2}{s}\mathcal{L}\{t\} = \frac{2}{s^3}.$$

$$\boxed{\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \; ; \quad n = 1, 2, 3, \dots \quad (s > 0)}$$

Proof (by induction):

Statement (n): $\mathcal{L}\{t^n\} = \dfrac{n!}{s^{n+1}}$ $\qquad n = 1, 2, 3, \dots$

① The Statement is True for $n=1$ - proved earlier, and also for $n=2$.

② Induction:

> Suppose Statement (n) is true.
> Show that Statement (n+1) is then also true.

$$n=1 \Rightarrow n=2 \Rightarrow n=3 \Rightarrow n=4 \Rightarrow \dots\dots$$

True

So: Suppose $\mathcal{L}\{t^n\} = \dfrac{n!}{s^{n+1}}$ is true. Then Look at $\mathcal{L}\{t^{n+1}\}$:

$$\mathcal{L}\{t^{n+1}\} = \int_0^\infty e^{-st} \cdot t^{n+1}\, dt$$

$$= \underbrace{-\frac{t^{n+1}}{s} e^{-st} \Big|_{t=0}^{\infty}}_{0 \text{ if } s>0} + \frac{n+1}{s} \underbrace{\int_0^\infty t^n e^{-st}\, dt}_{\mathcal{L}\{t^n\}}$$

$$\int e^{-st} \cdot t^{n+1}\, dt$$

$$u = t^{n+1}; \quad dv = e^{-st}$$

$$du = (n+1)t^n; \quad v = \frac{-1}{s} e^{-st}$$

$$= -\frac{t^{n+1}}{s} e^{-st} + \frac{n+1}{s} \int t^n e^{-st} dt$$

$$\Rightarrow \mathcal{L}\{t^{n+1}\} = \frac{n+1}{s} \cdot \underbrace{\mathcal{L}\{t^n\}}_{= \frac{n!}{s^{n+1}} \text{ by assumption}} = \frac{n+1}{s} \frac{n!}{s^{n+1}} = \frac{(n+1)!}{s^{n+2}}$$

$$\Rightarrow \mathcal{L}\{t^{n+1}\} = \frac{(n+1)!}{s^{(n+1)+1}}$$ Statement (n+1) ✓ $\blacksquare$

Example 3: $\mathcal{L}\{e^{-3t}\}$

$s > -3$ (otherwise it diverges)

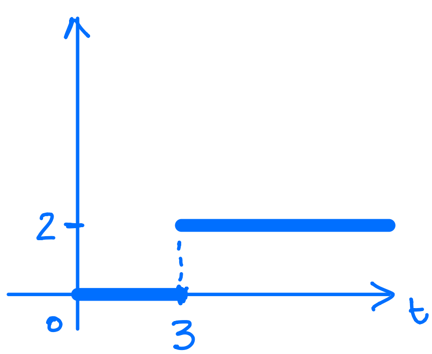
$$F(s) = \mathcal{L}\{e^{-3t}\} = \int_0^\infty e^{-st}\cdot e^{-3t}\,dt = \int_0^\infty e^{-(s+3)t}\,dt$$

$f(t) = e^{-3t}$

$$= \frac{-1}{s+3}\, e^{-(s+3)t}\Big|_{t=0}^{\infty} = 0 - \frac{-1}{s+3} = \frac{1}{s+3}$$

Example 4: A step function:

$$f(t) = \begin{cases} 0, & \text{if } 0 \le t < 3 \\ 2, & \text{if } 3 \le t < \infty \end{cases}$$



$$\boxed{F(s) = \frac{2}{s} e^{-3s};\ s > 0}$$

$$F(s) = \int_0^\infty e^{-st}\cdot f(t)\,dt$$

$$= \underbrace{\int_0^3 e^{-st}\cdot 0\,dt}_{0} + \int_3^\infty e^{-st}\cdot 2\,dt$$

$$= 2\int_3^\infty e^{-st}\,dt$$

$$= 2\,\frac{-1}{s}\, e^{-st}\Big|_{t=3}^{\infty}$$

$$= 0 - \frac{-2}{s} e^{-3s} \qquad (s > 0)$$

$$= \frac{2}{s} e^{-3s}$$

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} e^{at}\,dt = \int_0^\infty e^{(-s+a)t}\,dt$$

$$= \frac{1}{-s+a} e^{(-s+a)t}\Big|_{t=0}^{\infty}$$

The integral converges iff $-s+a<0 \iff s>a$

and then it equals:

$$0 - \frac{1}{-s+a}\cdot 1 = \frac{1}{s-a}$$

$$\boxed{\mathcal{L}\{e^{at}\} = \frac{1}{s-a}\,;\ s>a}$$

$$\mathcal{L}\{\sin(at)\} = \int_0^\infty e^{-st}\sin(at)\,dt$$

$\int e^{-st}\sin(at)\,dt$

$u=\sin(at)$; $dv=e^{-st}$

$du=a\cos(at)$; $v=\frac{-1}{s}e^{-st}$

$$= \underbrace{\frac{-1}{s}e^{-st}\sin(at)\Big|_0^\infty}_{0-0=0} - \int_0^\infty \frac{-1}{s}e^{-st}\cdot a\cos(at)\,dt$$

$s>0$:

$$= \frac{a}{s}\int_0^\infty e^{-st}\cdot\cos(at)\,dt \rightsquigarrow \boxed{\mathcal{L}\{\sin(at)\} = \frac{a}{s}\mathcal{L}\{\cos(at)\}}$$

$$= \frac{a}{s}\left(\underbrace{\frac{-1}{s}e^{-st}\cos(at)\Big|_{t=0}^{\infty}}_{0-\frac{-1}{s}} - \int_0^\infty \frac{+1}{s}e^{-st}a\sin(at)\,dt\right)$$

$\int e^{-st}\cos(at)\,dt$

$u=\cos(at)$; $dv=e^{-st}$

$du=-a\sin(at)$; $v=\frac{-1}{s}e^{-st}$

$$= \frac{a}{s}\left(\frac{1}{s} - \frac{a}{s}\mathcal{L}\{\sin(at)\}\right)$$

$$\Rightarrow \mathcal{L}\{\sin(at)\} = \frac{a}{s^2} - \frac{a^2}{s^2}\mathcal{L}\{\sin(at)\}$$

$$\frac{s^2+a^2}{s^2}\mathcal{L}\{\sin(at)\} = \frac{a}{s^2} \quad \Big| \cdot \frac{s^2}{s^2+a^2}$$

$$\boxed{\mathcal{L}\{\sin(at)\} = \frac{a}{s^2+a^2} \; ; \; s>0}$$

$$\boxed{\mathcal{L}\{\sin(at)\} = \frac{a}{s}\mathcal{L}\{\cos(at)\}}$$

$$\Rightarrow \mathcal{L}\{\cos(at)\} = \frac{s}{a}\mathcal{L}\{\sin(at)\} = \frac{s}{a} \cdot \frac{a}{s^2+a^2}$$

$$\boxed{\mathcal{L}\{\cos(at)\} = \frac{s}{s^2+a^2} \; ; \; s>0}$$

Recall hyperbolic functions:

$$\sinh(x) = \frac{e^{x} - e^{-x}}{2} \; ; \; \cosh(x) = \frac{e^{x} + e^{-x}}{2}$$

$$\mathcal{L}\{\sinh(at)\} = \mathcal{L}\left\{\frac{e^{at} - e^{-at}}{2}\right\} = \frac{1}{2}\underbrace{\mathcal{L}\{e^{at}\}}_{\frac{1}{s-a},\; s>a} - \frac{1}{2}\underbrace{\mathcal{L}\{e^{-at}\}}_{\frac{1}{s+a},\; s>-a}$$

$$= \frac{1}{2}\left(\frac{1}{s-a} - \frac{1}{s+a}\right) = \frac{1}{2} \; \frac{\cancel{s}+a-\cancel{s}+a}{s^2-a^2} = \frac{a}{s^2-a^2}$$

$$\boxed{\mathcal{L}\{\sinh(at)\} = \frac{a}{s^2-a^2} \; ; \; s>|a|}$$

Similarly:

$$\boxed{\mathcal{L}\{\cosh(at)\} = \frac{s}{s^2-a^2} \; ; \; s>|a|}$$