Properties of the Laplace Transform

I) Translation Theorem

Suppose $f(t)$ is some function w/ Laplace transform

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

Consider the Laplace transform of $e^{at} \cdot f(t)$ for some $a \in \mathbb{R}$:

$$\mathcal{L}\{e^{at} f(t)\} = \int_0^\infty e^{-st} \cdot e^{at} f(t)\,dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t)\,dt$$

$$= F(s-a)$$

=> Translation theorem:

$$\boxed{\mathcal{L}\{e^{at} f(t)\} = F(s-a) = \mathcal{L}\{f(t)\}_{s \to s-a}}$$

1) $$\mathcal{L}\{e^{t} \sin(4t)\} = \mathcal{L}\{\sin(4t)\}_{s \to s-1}$$

$$= \left.\frac{4}{s^2+16}\right|_{s \to s-1} = \frac{4}{(s-1)^2+16}$$

Freq. domain? $s > 0 \rightarrow (s-1) > 0 \rightarrow s > 1$

(2) $\mathcal{L}\{e^{-2t}\cos^2(3t)\} = \mathcal{L}\{\cos^2(3t)\}_{s \to s+2}$

$$= \frac{1}{2}\frac{1}{s+2} + \frac{1}{2}\frac{s+2}{(s+2)^2+36} \; ; \quad s > -2$$

Freq. domain: $s>0 \mapsto s+2>0 \mapsto s>-2$

$\mathcal{L}\{\cos^2(3t)\} = ?$ Use double-angle formula:

$$\cos^2\theta = \frac{1+\cos(2\theta)}{2}$$

$$= \mathcal{L}\left\{\frac{1+\cos(2\cdot 3t)}{2}\right\} = \mathcal{L}\left\{\frac{1}{2}+\frac{1}{2}\cos(6t)\right\}$$

$$= \frac{1}{2}\cdot\frac{1}{s} + \frac{1}{2}\cdot\frac{s}{s^2+36}$$

$(s>0) \qquad (s>0)$

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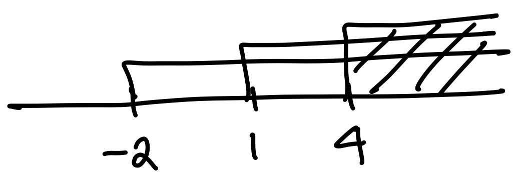
(3) $\mathcal{L}\{t(e^{-t}-e^{2t})^2\} = \mathcal{L}\{t(e^{-2t}-2e^{t}+e^{4t})\}$

$$= \mathcal{L}\{e^{-2t}\cdot t\} - 2\mathcal{L}\{e^{t}\cdot t\} + \mathcal{L}\{e^{4t}\cdot t\}$$

$$= \mathcal{L}\{t\}\Big|_{s\to s+2} - 2\mathcal{L}\{t\}_{s\to s-1} + \mathcal{L}\{t\}_{s\to s-4}$$

$$= \frac{1}{(s+2)^2} - 2\cdot\frac{1}{(s-1)^2} + \frac{1}{(s-4)^2} \qquad s>4$$

$s>-2 \qquad s>1 \qquad s>4$



II Derivatives of Laplace Transforms:

Let $F(s) = \mathcal{L}\{f(t)\}$ be the Laplace transform of some function.

$$\frac{d}{ds}F(s) = \frac{d}{ds}\int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^\infty \frac{\partial}{\partial s}\left[e^{-st} f(t)\right] dt$$

$$= \int_0^\infty e^{-st}(-t)f(t)\,dt = -\mathcal{L}\{t f(t)\}$$

$$\Rightarrow \boxed{\mathcal{L}\{t f(t)\} = -\frac{d}{ds}F(s)} = -\frac{d}{ds}\left[\mathcal{L}\{f(t)\}\right]$$

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④ $\mathcal{L}\{te^{2t}\} = -\frac{d}{ds}\mathcal{L}\{e^{2t}\} = -\frac{d}{ds}\frac{1}{s-2} = \frac{1}{(s-2)^2}$

$(s>2)$

Try 2^{nd} derivative?

$$\frac{d}{ds}\left[\frac{d}{ds}F(s) = -\int_0^\infty e^{-st}\cdot t f(t)\,dt\right]$$

$$\frac{d^2}{ds^2}F(s) = -\int_0^\infty \frac{\partial}{\partial s} e^{-st}\, t f(t)\,dt$$

$$= -\int_0^\infty e^{-st}\cdot(-t)\cdot t f(t)\,dt$$

$$= +\int_0^\infty e^{-st}\cdot\left(t^2 f(t)\right) dt = +\mathcal{L}\{t^2 f(t)\}$$

$$\Rightarrow \boxed{\mathcal{L}\{t^2 f(t)\} = +\frac{d^2}{ds^2}F(s) = \frac{d^2}{ds^2}\left[\mathcal{L}\{f(t)\}\right]}$$

Generally:

$$\boxed{\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s) = (-1)^n \frac{d^n}{ds^n}\left[\mathcal{L}\{f(t)\}\right]}$$

5) $\mathcal{L}\{t^2 \sin(5t)\} = \frac{d^2}{ds^2}\mathcal{L}\{\sin(5t)\}$

$$= \frac{d^2}{ds^2}\left(\frac{5}{s^2+25}\right) \qquad (s>0)$$

$$= \frac{d}{ds}\left(\frac{-5}{(s^2+25)^2}\cdot(+2s)\right)$$

$$= \frac{d}{ds}\quad \frac{-10s}{(s^2+25)^2}$$

$$= \frac{-10(s^2+25)^2 + 10s\cdot 2(s^2+25)\cdot 2s}{(s^2+25)^4}$$

$$= \frac{-10(s^2+25)+40s^2}{(s^2+25)^3} = \frac{30s^2-250}{(s^2+25)^3}$$

6) $\mathcal{L}\{t^4 e^{2t}\} = \frac{d^4}{ds^4}\mathcal{L}\{e^{2t}\}$ → True, but it will be much easier to use the translation property instead!

$$\| $$

$$\mathcal{L}\{t^4\}_{s\to s-2} = \left.\frac{4!}{s^5}\right|_{s\to s-2} = \frac{4!}{(s-2)^5} \quad s>2$$

$s>0 \to s-2>0 \to s>2$

III Laplace transforms of derivatives

$\int f'g = fg - \int fg'$
Integration by parts.

We saw last Thursday:

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt$$
$$= e^{-st} f(t)\Big|_{t=0}^{\infty} - \int_0^\infty -s e^{-st} f(t)\,dt$$
$$= -e^0 f(0) + s \underbrace{\int_0^\infty e^{-st} f(t)\,dt}_{\mathcal{L}\{f(t)\} = F(s)}$$

$$\boxed{\mathcal{L}\{f'(t)\} = sF(s) - f(0)}$$

$$\mathcal{L}\{f''(t)\} = \int_0^\infty e^{-st} f''(t)\,dt$$
$$= \int_0^\infty e^{-st} \left(f'(t)\right)'\,dt$$
$$= e^{-st} f'(t)\Big|_{t=0}^{\infty} - \int_0^\infty -s e^{-st} f'(t)\,dt$$
$$= -e^0 \cdot f'(0) + s \underbrace{\int_0^\infty e^{-st} f'(t)\,dt}_{\mathcal{L}\{f'(t)\} = sF(s) - f(0)}$$
$$= -f'(0) + s\left(sF(s) - f(0)\right)$$

$$\boxed{\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)}$$

$$\mathcal{L}\{f'''(t)\} = s^3F(s) - s^2f(0) - sf'(0) - f''(0)$$

$$\vdots$$

Generally:

$$\boxed{\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - s f^{(n-2)}(0) - f^{(n-1)}(0)}$$

This is the property crucial to solving ODE's with $\mathcal{L}$:

7 $y'' - 2y' + y = 0$ $\qquad y(0)=0;\ y'(0)=1$

$$\underbrace{\mathcal{L}\{y'' - 2y' + y\}}_{\mathcal{L}\{y''\} - 2\mathcal{L}\{y'\} + \mathcal{L}\{y\}} = \underbrace{\mathcal{L}\{0\}}_{0}$$

$\mathcal{L}\{y''\} = s^2 Y(s) - s\underbrace{y(0)}_{0} - \underbrace{y'(0)}_{1}$

$-2\mathcal{L}\{y'\} = -2(sY(s) - \underbrace{y(0)}_{0})$

$\mathcal{L}\{y\} = Y(s)$

$s^2 Y(s) - 1 - 2sY(s) + Y(s)$

$= (s^2 - 2s + 1)Y(s) - 1 \quad \Rightarrow \quad (s^2 - 2s + 1)Y(s) - 1 = 0$

$$Y(s) = \frac{1}{s^2 - 2s + 1} = \frac{1}{(s-1)^2}$$

$$Y(s) = \frac{1}{(s-1)^2} \Rightarrow y(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\} = te^t$$

$$= \mathcal{L}^{-1}\left\{-\left(\frac{1}{s-1}\right)'\right\}$$

Whose Laplace transform is $\frac{1}{(s-1)^2} = -\left(\frac{1}{s-1}\right)' = -F'(s)$

Recall $\mathcal{L}\{tf(t)\} = -F'(s)$

Whose Laplace transform is $\frac{1}{s-1}$? e^t!

$$\Rightarrow \mathcal{L}^{-1}\left\{-\left(\frac{1}{s-1}\right)'\right\} = te^t \Rightarrow \boxed{y(t) = te^t}$$

Solution to the ODE!

OR: Whose Laplace transform is $\frac{1}{s^2}$? t

How to shift from $\frac{1}{s^2}$ to $\frac{1}{(s-1)^2}$? Multiply by e^t

$$\Rightarrow \frac{1}{(s-1)^2} = \mathcal{L}\{te^t\}.$$

8 $y''(t) + y(t) = \cos(2t)$; $y(0)=0$, $y'(0)=1$.

The Old Way: $m^2+1=0$, $m=\pm 1$ $\quad y_c = C_1\cos(t) + C_2\sin(t)$

Undetermined Coefficients to find y_p:

$$y_p = A\cos(2t) + B\sin(2t)$$

$$y_p' = -2A\sin(2t) + 2B\cos(2t)$$

$$y_p'' = -4A\cos(2t) - 4B\sin(2t)$$

Plug back in:

$$-4A\cos(2t) - 4B\sin(2t) + A\cos(2t) + B\sin(2t) = \cos(2t)$$

$$-3A\cos(2t) - 3B\sin(2t) = \cos(2t) \Rightarrow \boxed{A = -1/3 \,;\, B = 0}$$

$\Rightarrow y_p = -\frac{1}{3}\cos(2t)$

$\Rightarrow$ General Solution: $y = c_1\cos(t) + c_2\sin(t) - \frac{1}{3}\cos(2t)$

Solve IVP: $\Rightarrow y' = -c_1\sin(t) + c_2\cos(t) + \frac{2}{3}\sin(2t)$

$y(0)=0 \Rightarrow c_1 - \frac{1}{3} = 0 \Rightarrow c_1 = \frac{1}{3}$

$y'(0) = 1 \Rightarrow c_2 = 1$

$\Rightarrow$ Solution: $\boxed{y = \frac{1}{3}\cos(t) + \sin(t) - \frac{1}{3}\cos(2t)}$

Remark: There are several steps that were needed here:

- ~ find the complementary solution y_c
- ~ find a particular solution y_p
- ~ write the general solution
- ~ solve the IVP

The Laplace way: Take Laplace of everything:

$$y''(t) + y(t) = \cos(2t) \quad ; \; y(0)=0, \; y'(0)=1.$$

$$s^2Y(s) - s\underset{\overset{\parallel}{0}}{y(0)} - \underset{\overset{\parallel}{1}}{y'(0)} + Y(s) = \frac{s}{s^2+4} \qquad (s>0)$$

$$(s^2+1)Y(s) - 1 = \frac{s}{s^2+4}$$

$$(s^2+1)Y(s) = \frac{s}{s^2+4} + 1 \Rightarrow \boxed{Y(s) = \frac{s}{(s^2+4)(s^2+1)} + \frac{1}{s^2+1}}$$

The function w/ this Laplace transform is the solution $y(t)$!

$$y(t) = \underbrace{\mathcal{L}^{-1}\left\{\frac{s}{(s^2+4)(s^2+1)}\right\}}_{(*)} + \underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}}_{\sin(t)} = \boxed{\begin{array}{l}\frac{1}{3}\cos(t) \\ -\frac{1}{3}\cos(2t) \\ +\sin(t)\end{array}}$$

$(*)$ $\mathcal{L}^{-1}\left\{\frac{1/3\, s}{s^2+1} - \frac{1/3\, s}{s^2+4}\right\} = \frac{1}{3}\cos(t) - \frac{1}{3}\cos(2t)$

Same!

Partial Fractions:

$$\frac{s}{(s^2+4)(s^2+1)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+1} = \frac{-\frac{1}{3}s}{s^2+4} + \frac{\frac{1}{3}s}{s^2+1}$$

$$s = As^3 + As + Bs^2 + B + Cs^3 + 4Cs + Ds^2 + 4D$$

$$s = \underbrace{(A+C)}_{0}s^3 + \underbrace{(B+D)}_{0}s^2 + \underbrace{(A+4C)}_{1}s + \underbrace{(B+4D)}_{0}$$

$$\begin{cases} A+C=0 & A=-C \\ B+D=0 & B=-D \\ A+4C=1 & \Rightarrow 3C=1 \Rightarrow C=\frac{1}{3} \quad A=-\frac{1}{3} \\ B+4D=0 & 3D=0 \Rightarrow \boxed{B=D=0} \end{cases}$$