The Inverse Laplace Transform

- Laplace Transform: $\boxed{f(t)} \xrightarrow{\mathcal{L}} \boxed{F(s) = \mathcal{L}\{f(t)\}}$
- Inverse problem: given $F(s)$, find $f(t)$ such that $F(s) = \mathcal{L}\{f(t)\}$:

$$\Rightarrow f(t) = \mathcal{L}^{-1}\{F(s)\}$$

$$\boxed{f(t) = \mathcal{L}^{-1}\{F(s)\}} \xleftarrow{\mathcal{L}^{-1}} \boxed{F(s)}$$

Inverse Laplace transforms we already know:

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^n}\right\} = \frac{1}{(n-1)!}t^{n-1}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{\sin(at)}{a}$$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos(at)$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2-a^2}\right\} = \frac{\sinh(at)}{a}$$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh(at)$$

Some basic examples:

$$\mathcal{L}^{-1}\left\{\frac{1}{s^5}\right\} = \frac{t^4}{4!}$$

$$\mathcal{L}^{-1}\left\{\frac{3}{s}\right\} = 3$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} = e^{-2t}$$

$$\mathcal{L}^{-1}\left\{\frac{3}{s^2+64}\right\} = \frac{3}{8}\sin(8t)$$

$$\mathcal{L}^{-1}\left\{\frac{2s}{s^2+64}\right\} = 2\cos(8t)$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} = e^{3t}$$

$$\mathcal{L}^{-1}\left\{\frac{5}{s^2-16}\right\} = \frac{5}{4}\sinh(4t)$$

$$\mathcal{L}^{-1}\left\{\frac{\sqrt{2}s}{s^2-7}\right\} = \sqrt{2}\cosh(\sqrt{7}t)$$

$$\mathcal{L}^{-1}\left\{\frac{2s-1}{s^2+16}\right\} = \mathcal{L}^{-1}\left\{\frac{2s}{s^2+16}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2+16}\right\}$$
$$= 2\cos(4t) - \frac{1}{4}\sin(4t).$$

Recall the Translation Theorem for $\mathcal{L}$:

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a) = \mathcal{L}\{f(t)\}_{s\to s-a}$$

Take $\mathcal{L}^{-1}$ of all sides and obtain:

Translation Theorem for the Inverse Laplace Transform:

$$\boxed{\mathcal{L}^{-1}\{F(s-a)\} = \mathcal{L}^{-1}\{F(s)\big|_{s\to s-a}\} = e^{at}f(t)}$$

Examples:

1. $\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^3}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^3}\Big|_{s\to s+2}\right\} = e^{-2t}\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} = e^{-2t}\cdot\frac{1}{2}t^2.$

2. $\mathcal{L}^{-1}\left\{\frac{1}{s^2+6s+11}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+3)^2+2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+2}\Big|_{s\to s+3}\right\}$

$= e^{-3t}\mathcal{L}^{-1}\left\{\frac{1}{s^2+2}\right\}$

$= e^{-3t}\frac{\sin(\sqrt{2}t)}{\sqrt{2}}$

Completing the square

$s^2+\underbrace{6s}_{2\cdot 3\cdot s}+11$ needs a $+9$

$(s^2+2\cdot3\cdot s+3^2)+2$

$\mathcal{L}^{-1}$ Using Partial Fractions:

1. $\mathcal{L}^{-1}\left\{\frac{1}{(s-1)(s+2)(s+4)}\right\} = \mathcal{L}^{-1}\left\{\frac{A}{s-1}+\frac{B}{s+2}+\frac{C}{s+4}\right\}$

$= Ae^{t}+Be^{-2t}+Ce^{-4t}$

$= \boxed{\frac{1}{15}e^{t}-\frac{1}{6}e^{-2t}+\frac{1}{10}e^{-4t}}$

$1 = A(s+2)(s+4)+B(s-1)(s+4)+$
$+C(s-1)(s+2)$

$s=-2$: $1 = B(-3\cdot2) \Rightarrow B = -1/6$

$s=-4$: $1 = C(-5\cdot -2) \Rightarrow C = 1/10$

$s=1$: $1 = A(3\cdot5) \Rightarrow A = 1/15$

Laplace Transform & Linear ODEs w/ Constant Coefficients (IVPs!)

$$\boxed{\begin{array}{c} a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = g(t) \\ y(0) = c_0;\ y'(0) = c_1;\ \ldots;\ y^{(n-1)}(0) = c_{n-1} \\ (a_n, a_{n-1}, \ldots, a_1, a_0 \text{ are functions of } t \text{ only!}) \end{array}} \xrightarrow[\text{of both sides}]{\mathcal{L}} \boxed{\begin{array}{l} \text{Algebraic Equation,} \\ \text{solve for} \\ \quad Y(s) = \mathcal{L}\{y(t)\} \end{array}} \xrightarrow{\mathcal{L}^{-1}} \boxed{\begin{array}{c} \text{Solution} \\ y(t) \end{array}}$$

Uses the properties from last time:

$$\mathcal{L}\{y'\} = sY(s) - y(0)$$
$$\mathcal{L}\{y''\} = s^2 Y(s) - s\,y(0) - y'(0)$$
$$\mathcal{L}\{y'''\} = s^3 Y(s) - s^2 y(0) - s\,y'(0) - y''(0)$$
$$\vdots$$

Example: $y'' - 4y' + 4y = t^3 e^{2t}$
$y(0) = y'(0) = 0$

Old method: Undetermined Coefficients:

① y_c: $m^2 - 4m + 4 = 0$; $(m-2)^2 = 0$; $m_1 = m_2 = 2$

$$y_c = c_1 e^{2t} + c_2 t e^{2t}$$

② $y_p = (At^3 + Bt^2 + Ct + D)e^{2t} \cdot t^2$ (to avoid duplication)

$$= (At^5 + Bt^4 + Ct^3 + Dt^2)e^{2t}$$

$$y_p' = (5At^4 + 4Bt^3 + 3Ct^2 + 2Dt)e^{2t} + 2(At^5 + Bt^4 + Ct^3 + Dt^2)e^{2t}$$
$$= (2At^5 + (5A+2B)t^4 + (4B+2C)t^3 + (3C+2D)t^2 + 2Dt)e^{2t}$$

$$y_p'' = (10At^4 + (20A+8B)t^3 + (12B+6C)t^2 + (6C+4D)t + 2D)e^{2t}$$
$$+ e^{2t}(4At^5 + (10A+4B)t^4 + (8B+4C)t^3 + (6C+4D)t^2 + 4Dt)$$

$$= (4At^5 + (20A+4B)t^4 + (20A+16B+4C)t^3 + (12B+12C+4D)t^2 + (6C+8D)t + 2D)e^{2t}$$

Put y_p, y_p', y_p'' back in the original equation and find the coefficients:

t^5e^{2t}: $4A-8A+4A=0 \Rightarrow 0=0$ (no info)

t^4e^{2t}: $(20A+4B)-(20A+8B)+4B=0 \Rightarrow 0=0$ (no info)

t^3e^{2t}: $(20A+\cancel{16B}+4C)-(\cancel{16B}+8C)+4C=1$

$20A=1 \Rightarrow A=1/20$

t^2e^{2t}: $(12B+\cancel{12C}+4D)-(\cancel{12C}+8D)+4D=0$

$12B=0 \Rightarrow B=0$

te^{2t}: $(6C+\cancel{8D})-\cancel{8D}=0 \Rightarrow C=0$

e^{2t}: $2D=0 \Rightarrow D=0$

$$\Rightarrow \boxed{y_p=\frac{1}{20}t^5e^{2t}}$$

$\Rightarrow$ General Solution: $y=y_c+y_p=c_1e^{2t}+c_2te^{2t}+\frac{1}{20}t^5e^{2t}$

$y'=2c_1e^{2t}+c_2e^{2t}+2c_2te^{2t}+\frac{1}{4}t^4e^{2t}+\frac{1}{10}t^5e^{2t}$

Plug in IVP: $y(0)=0 \Rightarrow c_1=0$

$y'(0)=0 \Rightarrow 2c_1+c_2=0 \Rightarrow c_2=0$

$\Rightarrow$ IVP Solution: $\boxed{y=\frac{1}{20}t^5e^{2t}}$

$$y'' - 4y' + 4y = t^3 e^{2t}$$
$$y(0) = y'(0) = 0$$

Same problem w/ Laplace transform:

$$\underbrace{\mathcal{L}\{y''\} - 4\mathcal{L}\{y'\} + 4\mathcal{L}\{y\}} = \underbrace{\mathcal{L}\{t^3 e^{2t}\}}$$

$$s^2 Y(s) - s\overset{0}{\widehat{y(0)}} - \overset{0}{\widehat{y'(0)}}$$
$$-4sY(s) + 4\overset{0}{\widehat{y(0)}}$$
$$+4Y(s) = (s^2 - 4s + 4)Y(s)$$
$$= (s-2)^2 Y(s)$$

$$\mathcal{L}\{t^3\}\Big|_{s \to s-2} = \frac{3!}{s^4}\Big|_{s \to s-2} = \frac{6}{(s-2)^4}$$

$$\Rightarrow (s-2)^2 Y(s) = \frac{6}{(s-2)^4} \Rightarrow Y(s) = \frac{6}{(s-2)^6}$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\left\{\frac{6}{(s-2)^6}\right\} = 6\mathcal{L}^{-1}\left\{\frac{1}{s^6}\Big|_{s \to s-2}\right\} = 6e^{2t} \cdot \frac{t^5}{5!} = \frac{1}{20} t^5 e^{2t}$$

$$\boxed{y(t) = \frac{1}{20} t^5 e^{2t}}$$