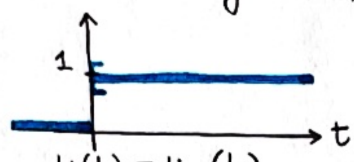
Laplace Transform & Unit Step Functions

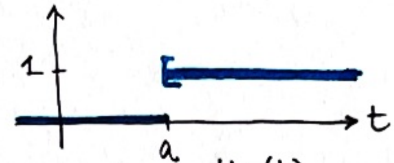
Def.: The **Unit Step Function** (aka the **Heaviside Function**): $\boxed{u(t)=\begin{cases}0, & t<0\\ 1, & t\geq 0\end{cases}}$

- models situations where a signal can be either "on" or "off"
- translations of $u(t)$ allows one to turn signals off at times other than 0:

$$\boxed{u_a(t)=\begin{cases}0, & \text{if } t<a\\ 1, & \text{if } t\geq a\end{cases}\quad (a>0)}$$



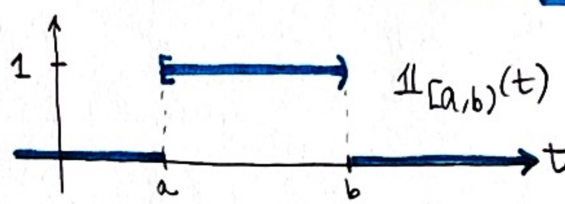
$u(t)=u_0(t)$



$u_a(t)$

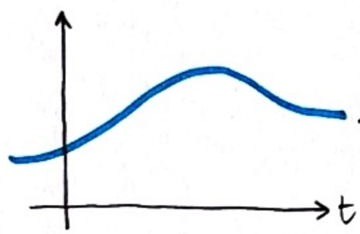
- We can turn a signal "on" at $t=a$, and "off" at $t=b$, using the **indicator function** $\boxed{\mathbb{1}_{[a,b)}}$

$$\boxed{\mathbb{1}_{[a,b)}(t)=u_a(t)-u_b(t)=\begin{cases}0, & t<a\\ 1, & a\leq t<b\\ 0, & t>b\end{cases}}$$

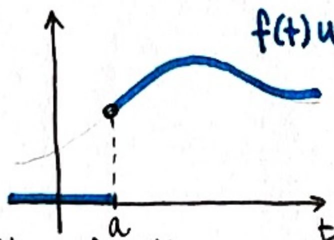


$\mathbb{1}_{[a,b)}(t)$

- Multiplying a function $f(t)$ by $u_a(t)$ or $\mathbb{1}_{[a,b)}(t)$ can "turn off" portions of the graph of f:

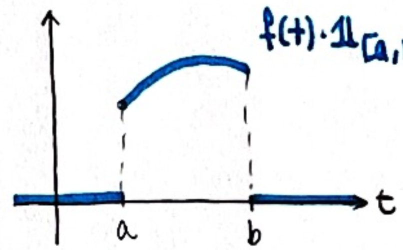


$f(t)$



$f(t)u_a(t)$

"turns off" the graph of f before a



$f(t)\cdot\mathbb{1}_{[a,b)}(t)$

"turns off" the graph of f everywhere outside $a\leq t<b$

- $\boxed{\mathcal{L}\{u_a(t)\}=\dfrac{e^{-as}}{s};\ s>0 \quad (a>0)}$

$$\mathcal{L}\{u_a(t)\}=\int_0^\infty e^{-st}u_a(t)\,dt$$
$$=\int_a^\infty e^{-st}\,dt=\frac{-1}{s}e^{-st}\Big|_{t=a}^{\infty}=\frac{1}{s}e^{-sa}$$

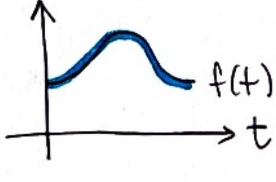
- **The Second Translation Theorem:**

$(a>0)$ $\boxed{\mathcal{L}\{f(t-a)u_a(t)\}=e^{-as}F(s)}$

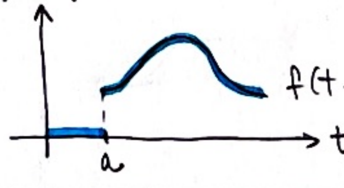
Inverse Form:

$\boxed{\mathcal{L}^{-1}\{e^{-as}F(s)\}=f(t-a)u_a(t)}$

Remark: What does $f(t-a)u_a(t)$ do? Say we have a function $f(t)$, defined on $t\in[0,\infty)$. Translating by $a>0$, i.e. $f(t-a)$, "shifts" the graph of f to the right by a units, and is now a function defined on $t\in[a,\infty)$. Multiplying by $u_a(t)$ does not change this, all it does is make the function defined on $[0,\infty)$, with the new function being 0 on $[0,a)$:



$f(t)$



$f(t-a)u_a(t)$