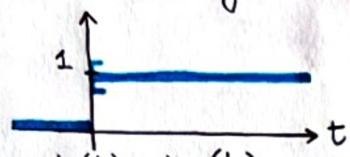
Laplace Transform & Unit Step Functions

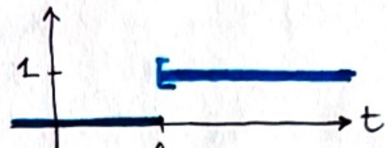
Def.: The Unit Step Function (aka the **Heaviside Function**): $\boxed{u(t) = \begin{cases} 0, & t<0 \\ 1, & t \geq 0 \end{cases}}$

- models situations where a signal can be either "on" or "off"
- Translations of $u(t)$ allows one to turn signals off at times other than 0:

$$\boxed{u_a(t) = \begin{cases} 0, & \text{if } t<a \\ 1, & \text{if } t \geq a \end{cases} \quad (a>0)}$$



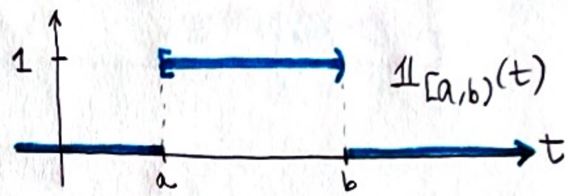
$u(t) = u_0(t)$



$u_a(t)$

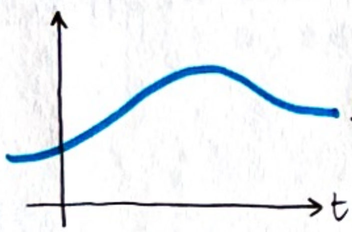
- We can turn a signal "on" at $t=a$, and "off" at $t=b$, using the **indicator function** $\boxed{\mathbb{1}_{[a,b)}}$

$$\boxed{\mathbb{1}_{[a,b)}(t) = u_a(t) - u_b(t) = \begin{cases} 0, & t<a \\ 1, & a \leq t < b \\ 0, & t > b \end{cases}}$$

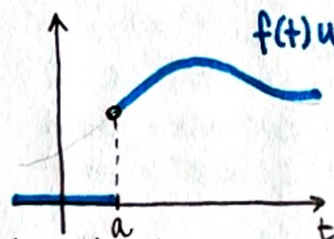


$\mathbb{1}_{[a,b)}(t)$

- Multiplying a function $f(t)$ by $u_a(t)$ or $\mathbb{1}_{[a,b)}(t)$ can "turn off" portions of the graph of f:

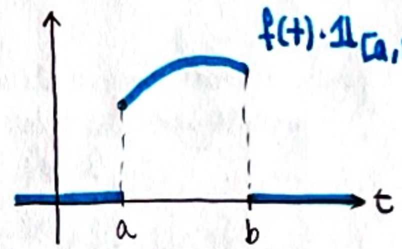


$f(t)$



$f(t)u_a(t)$

"turns off" the graph of f before a



$f(t) \cdot \mathbb{1}_{[a,b)}(t)$

"turns off" the graph of f everywhere outside $a \leq t < b$

- $\boxed{\mathcal{L}\{u_a(t)\} = \dfrac{e^{-as}}{s};\ s>0 \quad (a>0)}$

$$\mathcal{L}\{u_a(t)\} = \int_0^\infty e^{-st} u_a(t)\,dt = \int_a^\infty e^{-st}\,dt = \frac{-1}{s} e^{-st}\Big|_{t=a}^{\infty} = \frac{1}{s}e^{-sa}$$

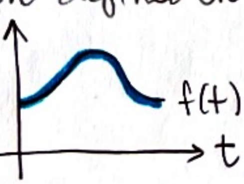
- **The Second Translation Theorem:**

$(a>0)$ $\boxed{\mathcal{L}\{f(t-a)u_a(t)\} = e^{-as}F(s)}$

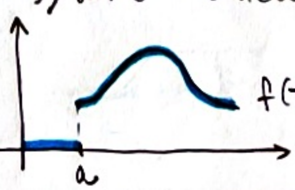
Inverse Form:

$\boxed{\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u_a(t)}$

Remark: What does $f(t-a)u_a(t)$ do? Say we have a function $f(t)$, defined on $t \in [0,\infty)$. Translating by $a>0$, i.e. $f(t-a)$, "shifts" the graph of f to the right by a units, and is now a function defined on $t \in [a,\infty)$. Multiplying by $u_a(t)$ does not change this, all it does is make the function defined on $[0,\infty)$, with the new function being 0 on $[0,a)$:



$f(t)$



$f(t-a)u_a(t)$

Proof of Second Translation Theorem:

$$\mathcal{L}\{f(t-a)u_a(t)\} = \int_0^\infty e^{-st} f(t-a)u_a(t)\,dt = \int_a^\infty e^{-st} f(t-a)\,dt$$

Change of variable: $u = t-a$, $du = dt$, $t=a \Rightarrow u=0$, $t\to\infty \Rightarrow u\to\infty$

$$= \int_0^\infty e^{-s(u+a)} f(u)\,du$$

$$= e^{-sa} \cdot \underbrace{\int_0^\infty e^{-su} \cdot f(u)\,du}_{= \mathcal{L}\{f(t)\} = F(s)}$$

$$\Rightarrow \boxed{\begin{aligned}\mathcal{L}\{f(t-a)u_a(t)\} &= e^{-as}F(s)\\ &= e^{-as}\mathcal{L}\{f(t)\}\end{aligned}}$$

① $f(t) = \begin{cases} (t-2)^3, & t \geq 2 \\ 0, & 0 \leq t < 2 \end{cases}$

Express in terms of step functions: $f(t) = (t-2)^3 u_2(t)$

$$\Rightarrow \mathcal{L}\{f(t)\} = \mathcal{L}\{(t-2)^3 u_2(t)\} = e^{-2s}\mathcal{L}\{t^3\} = e^{-2s} \cdot \frac{3!}{s^4}$$

② $f(t) = \begin{cases} (t-2)^3, & t \geq 3 \\ 0, & 0 \leq t < 3 \end{cases}$

$f(t) = (t-2)^3 u_3(t)$

The 2nd Translation Theorem does not immediately apply - the $u_3(t)$ suggests we need everything to be in terms of $(t-3)$, not $(t-2)$.

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{(t-2)^3 u_3(t)\} = \mathcal{L}\{(t-3+1)^3 u_3(t)\} = e^{-3s}\mathcal{L}\{(t+1)^3\}$$

$[(t-3)+1]^3$: what is the action on $(t-3)$? Add 1, cube everything

$\bullet \mapsto (\bullet+1)^3$

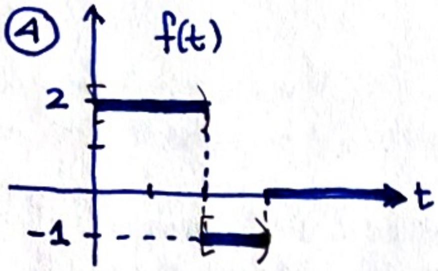
$$= e^{-3s}\mathcal{L}\{t^3 + 3t^2 + 3t + 1\}$$

$$= e^{-3s}\left(\frac{6}{s^4} + \frac{6}{s^3} + \frac{3}{s^2} + \frac{1}{s}\right).$$

③ $\mathcal{L}\{\sin(t)u_{2\pi}(t)\} = \mathcal{L}\{\sin(t-2\pi)u_{2\pi}(t)\} = e^{-2\pi s}\mathcal{L}\{\sin t\} = \dfrac{e^{-2\pi s}}{s^2+1}$

(sin is periodic w/ period 2π!)

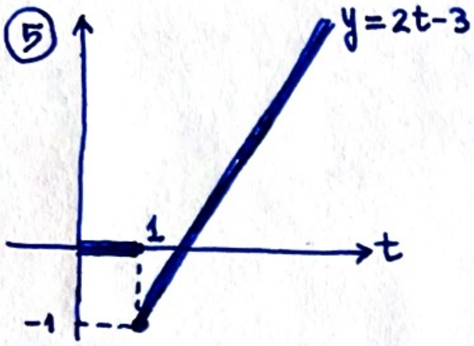
④



Express in terms of step functions:

$$f(t) = 2\cdot \mathbb{1}_{[0,2)}(t) - 1\cdot \mathbb{1}_{[2,3)}(t)$$
$$= 2(u_0(t) - u_2(t)) - (u_2(t) - u_3(t))$$
$$= 2u_0(t) - 3u_2(t) + u_3(t)$$
$$\Rightarrow \mathcal{L}\{f(t)\} = 2\mathcal{L}\{u_0(t)\} - 3\mathcal{L}\{u_2(t)\} + \mathcal{L}\{u_3(t)\}$$
$$= 2\cdot\frac{1}{s} - 3\cdot\frac{e^{-2s}}{s} + \frac{e^{-3s}}{s} \quad (s>0)$$

⑤



$f(t)$ is the line $2t-3$ "turned off" on $[0,1)$

$$f(t) = (2t-3)u_1(t)$$
$$\Rightarrow \mathcal{L}\{f(t)\} = \mathcal{L}\{\underbrace{(2t-3)}_{\text{needs to be in terms of } (t-1)}u_1(t)\}$$
$$= \mathcal{L}\{(2(t-1)-1)u_1(t)\}$$
$$= e^{-s}\cdot\mathcal{L}\{2t-1\}$$
$$= e^{-s}\cdot\left(2\cdot\frac{1}{s^2} - \frac{1}{s}\right)$$
$$= e^{-s}\left(\frac{2}{s^2} - \frac{1}{s}\right); \quad (s>0)$$

$\rightsquigarrow \mathcal{L}\{t\,u_2(t)\} = \mathcal{L}\{[(t-2)+2]u_2(t)\} = e^{-2s}\mathcal{L}\{t+2\} = e^{-2s}\left(\frac{1}{s^2} + \frac{2}{s}\right). \quad (s>0)$

$\rightsquigarrow \mathcal{L}\{(t-1)^3 e^{t-1} u_1(t)\} = e^{-s}\mathcal{L}\{t^3 e^t\} = e^{-s}\left(\mathcal{L}\{t^3\}\Big|_{s\to s-1}\right) = e^{-s}\cdot\left(\frac{3!}{s^4}\Big|_{s\to s-1}\right)$

$$= e^{-s}\,\frac{3!}{(s-1)^4} \quad (s>1)$$

2nd Translation Thm.

1st Translation Thm.

⑥ $\mathcal{L}^{-1}\left\{\frac{e^{-\pi s/2}}{s^2+9}\right\} = \mathcal{L}^{-1}\left\{e^{-\frac{\pi}{2}s}\cdot\frac{1}{s^2+9}\right\} = u_{\pi/2}(t)\cdot\mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\}\Big|_{t\mapsto t-\pi/2}$

$$= u_{\pi/2}(t)\cdot\frac{1}{3}\sin(3t)\Big|_{t\mapsto t-\pi/2}$$
$$= \frac{1}{3}\sin\left(3t - \frac{3\pi}{2}\right)u_{\pi/2}(t)$$
$$\left[= \frac{1}{3}\cos(3t)\,u_{\pi/2}(t)\right]$$

Inverse Form of 2nd Translation:

$$\boxed{\mathcal{L}^{-1}\{e^{-as}F(s)\} = u_a(t)\,\mathcal{L}^{-1}\{F(s)\}\Big|_{t\to t-a}}$$

$\left[\sin(3t - 3\pi/2) = \sin(3t)\underbrace{\cos(3\pi/2)}_{0} - \cos(3t)\underbrace{\sin(3\pi/2)}_{-1}\right.$
$= \cos(3t)$

Laplace Transform of Periodic Functions

Say we have a function f with period $T>0$, i.e. $f(t)=f(t+T),\ \forall t\geq 0$.
Laplace transforms of such functions can be obtained by <u>integration over 1 period</u>:

> **Suppose $f(t)$ is piecewise continuous on $t\in[0,\infty)$, is of exponential order, and is periodic with period T. Then:**
>
> $$\boxed{\mathcal{L}\{f(t)\}=\frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)\,dt}=\frac{1}{1-e^{-sT}}\mathcal{L}\{f(t)\underbrace{(1-u_T(t))}_{\mathbb{1}_{[0,T)}}\}$$

<u>Proof</u>: $\mathcal{L}\{f(t)\}=\int_0^\infty e^{-st}f(t)\,dt$

$$=\int_0^T e^{-st}f(t)\,dt+\int_T^\infty e^{-st}f(t)\,dt$$

$t=u+T$, $dt=du$

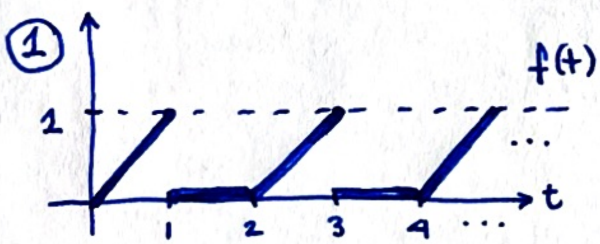
$$\int_T^\infty e^{-st}f(t)\,dt=\int_0^\infty e^{-s(u+T)}\underbrace{f(u+T)}_{=f(u)}\,du=e^{-sT}\underbrace{\int_0^\infty e^{-su}f(u)\,du}_{\mathcal{L}\{f(t)\}}$$

$$\Rightarrow \mathcal{L}\{f(t)\}=\int_0^T e^{-st}f(t)\,dt+e^{-sT}\mathcal{L}\{f(t)\}$$

$$(1-e^{-sT})\mathcal{L}\{f(t)\}=\int_0^T e^{-st}f(t)\,dt$$

$$\mathcal{L}\{f(t)\}=\frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)\,dt. \quad /\!/ \quad \blacksquare$$

①



Period: $T=2$
Repeating sequence:

$$\mathcal{L}\{f(t)\}=\frac{1}{1-e^{-2s}}\int_0^2 e^{-st}f(t)\,dt$$
$$=\frac{1}{1-e^{-2s}}\int_0^1 e^{-st}\cdot t\,dt$$
$$=\frac{1}{1-e^{-2s}}\mathcal{L}\{t(1-u_1(t))\}$$
$$=\frac{1}{1-e^{-2s}}\Big(\underbrace{\mathcal{L}\{t\}}_{\frac{1}{s^2}}-\underbrace{\mathcal{L}\{t\,u_1(t)\}}_{}\Big)$$

$$\mathcal{L}\{((t-1)+1)u_1(t)\}=e^{-s}\mathcal{L}\{t+1\}=e^{-s}\left(\frac{1}{s^2}+\frac{1}{s}\right)$$

$$=\frac{1}{1-e^{-2s}}\left(\frac{1}{s^2}-\frac{e^{-s}}{s^2}-\frac{e^{-s}}{s}\right)$$
$$=\frac{1-e^{-s}-se^{-s}}{(1-e^{-2s})s^2} \quad /\!/$$

OR INTEGRATE DIRECTLY

by parts

$$\int e^{-st}\cdot t\,dt$$
$$=\frac{-1}{s}e^{-st}\cdot t+\frac{1}{s}\int e^{-st}\,dt$$
$$=\frac{-1}{s}e^{-st}t-\frac{1}{s^2}e^{-st}$$
$$\Rightarrow \int_0^1 e^{-st}\cdot t\,dt=\frac{-1}{s}e^{-s}-\frac{1}{s^2}e^{-s}+\frac{1}{s^2}$$