Convolution & the Laplace Transform

DEF.: For two piecewise continuous functions f and g on $[0,\infty)$, the convolution of f & g is a new function $(f*g)(t)$, defined by:

$$\boxed{(f*g)(t) := \int_0^t f(t-y)g(y)\,dy}$$

Example: Take $f(t)=e^{2t}$ and $g(t)=e^{-t}$:

$$\begin{aligned}(f*g)(t) = \int_0^t e^{2(t-y)}e^{-y}\,dy &= \int_0^t e^{2t}e^{-3y}\,dy \\ &= e^{2t}\cdot\int_0^t e^{-3y}\,dy \\ &= e^{2t}\cdot \frac{-1}{3}e^{-3y}\Big|_{y=0}^{t} \\ &= e^{2t}\left(-\frac{1}{3}e^{-3t}+\frac{1}{3}\right)\end{aligned}$$

$$(f*g)(t) = \frac{-1}{3}e^{-t}+\frac{1}{3}e^{2t}$$

Lemma: Convolution is commutative, i.e. $(f*g)(t)=(g*f)(t)$.

Pf.: $(f*g)(t) = \int_0^t f(t-y)g(y)\,dy$

$$\begin{aligned}(g*f)(t) &= \int_0^t g(t-y)f(y)\,dy \\ &= \int_t^0 g(x)f(t-x)(-dx) \\ &= \int_0^t g(x)f(t-x)\,dx \\ &= (f*g)(t).\end{aligned}$$

Change of variable:

$x=t-y$ $\qquad$ $y=0 \Rightarrow x=t$

$dx=-dy$ $\qquad$ $y=t \Rightarrow x=0$

$\blacksquare$

Connection to Laplace Transform:

Suppose f, g have Laplace transforms $F(s) = \mathcal{L}\{f(t)\}$ & $G(s) = \mathcal{L}\{g(t)\}$.
Look at their product:

$$F(s)\cdot G(s) = \mathcal{L}\{f(t)\}\cdot\mathcal{L}\{g(t)\}$$

$$= \int_0^\infty e^{-sz} f(z)\,dz \cdot \int_0^\infty e^{-sy} g(y)\,dy$$

$$= \int_0^\infty \int_0^\infty e^{-sz} e^{-sy} f(z) g(y)\,dz\,dy$$

$$= \int_0^\infty g(y) \Big(\underbrace{\int_0^\infty e^{-s(z+y)} f(z)\,dz}_{} \Big)\,dy$$

$z = t - y$ (temporarily fix y) (new variable: t)

$dz = dt$

$z = 0 \Rightarrow t = y$; $z \to \infty \Rightarrow t \to \infty$

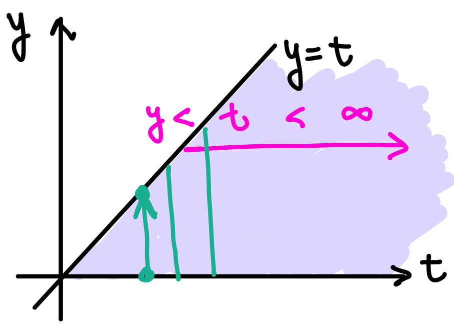
$$\int_y^\infty e^{-st} f(t-y)\,dt$$

$$= \int_0^\infty g(y) \Big(\int_y^\infty e^{-st} f(t-y)\,dt \Big)\,dy$$

$$= \underbrace{\int_0^\infty \int_y^\infty g(y) f(t-y) e^{-st}\,dt\,dy}_{}$$

double integral over region (t, y)

$$\begin{cases} 0 < y < \infty \\ y < t < \infty \end{cases}$$



Reverse order of integration:

$$\int_0^\infty \int_0^t dy\,dt$$

t goes from $0 \to \infty$

y goes from 0 to t

$$F(s)G(s) = \int_0^\infty \int_0^t g(y) e^{-st} f(t-y)\,dy\,dt$$

$$= \int_0^\infty e^{-st} \Big(\underbrace{\int_0^t g(y) f(t-y)\,dy}_{(f*g)(t)} \Big)\,dt = \int_0^\infty e^{-st} (f*g)(t)\,dt$$

$$= \mathcal{L}\{f*g\}.$$

We just proved:

$$\boxed{\mathcal{L}\{f*g\} = F(s)\,G(s) = \mathcal{L}\{f(t)\}\,\mathcal{L}\{g(t)\}}$$

Inverse form:

$$\boxed{\mathcal{L}^{-1}\{F(s)G(s)\} = (f*g)(t)}$$

This is VERY useful because it allows one to find inverse Laplace transforms of products $F(s)G(s)$ which are otherwise difficult to handle:

Example: $\mathcal{L}^{-1}\left\{\frac{1}{(s^2+4)^2}\right\} = \mathcal{L}^{-1}\{F(s)\,G(s)\}$ where $F(s) = G(s) = \frac{1}{s^2+4}$

$= (f*g)(t)$ $\Rightarrow f(t) = g(t) = \frac{1}{2}\sin(2t)$

$$(f*g)(t) = \int_0^t \frac{1}{2}\sin\big(2(t-y)\big)\,\frac{1}{2}\sin(2y)\,dy$$

$$= \frac{1}{4}\int_0^t \sin(2t-2y)\sin(2y)\,dy$$

$\sin A \sin B = \frac{1}{2}\big(\cos(A-B) - \cos(A+B)\big)$

$$= \frac{1}{4}\int_0^t \frac{1}{2}\Big(\cos(2t-4y) - \cos(2t)\Big)\,dy$$

$$= \frac{1}{8}\left[\int_0^t \cos(2t-4y)\,dy - \int_0^t \cos(2t)\,dy\right]$$

$$= \frac{1}{8}\left[\frac{-1}{4}\sin(2t-4y)\Big|_{y=0}^{t} - \cos(2t)\cdot y\Big|_{y=0}^{t}\right]$$

$$= \frac{1}{8}\left[\frac{-1}{4}\underbrace{\sin(-2t)}_{=-\sin(2t)} + \frac{1}{4}\sin(2t) - \cos(2t)\cdot t\right]$$

$$= \frac{1}{8}\left[\frac{1}{2}\sin(2t) - t\cos(2t)\right] = \boxed{\frac{1}{16}\sin(2t) - \frac{1}{8}t\cos(2t)}$$

Example: $4y'' + y = g(t)$; $y(0)=3,\ y'(0)=-7$

$$\mathcal{L}\{4y''\} + \mathcal{L}\{y\} = \mathcal{L}\{g(t)\}$$

$$4\big(s^2Y(s) - s\underbrace{y(0)}_{3} - \underbrace{y'(0)}_{-7}\big) + Y(s) = G(s)$$

$$4s^2Y(s) - 12s + 28 + Y(s) = G(s)$$

$$(4s^2+1)Y(s) = G(s) + 12s - 28 \Rightarrow Y(s) = \frac{G(s)}{4s^2+1} + \frac{12s-28}{4s^2+1}$$

$$\Rightarrow y(t) = \underbrace{\mathcal{L}^{-1}\left\{\frac{G(s)}{4s^2+1}\right\}} + \underbrace{\mathcal{L}^{-1}\left\{\frac{12s-28}{4s^2+1}\right\}}$$

$$\frac{12}{4}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1/4}\right\} - \frac{28}{4}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1/4}\right\}$$

$$= 3\cdot\cos\left(\tfrac{1}{2}t\right) - 7\cdot\frac{1}{1/2}\sin\left(\tfrac{1}{2}t\right)$$

$$= 3\cos(t/2) - 14\sin(t/2)$$

$$\mathcal{L}^{-1}\{G(s)\cdot F(s)\}$$

$$= (f*g)(t),$$

where $F(s) = \dfrac{1}{4s^2+1}$

$$\Rightarrow f(t) = \mathcal{L}^{-1}\left\{\frac{1}{4s^2+1}\right\}$$

$$= \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1/4}\right\}$$

$$= \frac{1}{4}\cdot\frac{1}{1/2}\sin\left(\tfrac{1}{2}t\right)$$

$$= \frac{1}{2}\sin\left(\tfrac{1}{2}t\right)$$

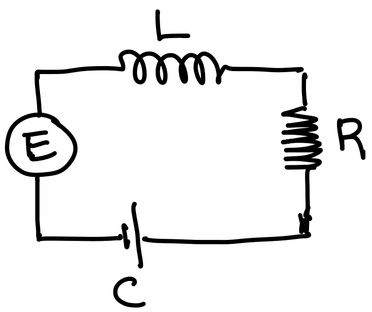
HUGE ADVANTAGE: This gives a general formula for solutions to this equation, which can be used for any appropriate g!

Solution to ODE $\Longrightarrow$

$$\boxed{y(t) = 3\cos\left(\tfrac{t}{2}\right) - 14\sin\left(\tfrac{t}{2}\right) + (f*g)(t)}$$

where $f(t) = \frac{1}{2}\sin\left(\frac{t}{2}\right)$ and $g(t)$ is the given forcing function.

Example: Kirchhoff's Second Law for a series RLC circuit:
(resistor, inductor, capacitor)



$$\boxed{L\frac{di}{dt} + R\cdot i(t) + \frac{1}{C}\int_0^t i(y)\,dy = E(t)}$$

where $E(t)$ = voltage, $i(t)$ = current.

HW: Find the current in such an RLC circuit, where

$L = 0.1h$; $R = 2\Omega$; $C = 0.1f$;
$i(0) = 0$

$E(t) = 120t\,(1 - u_1(t))$

Trick: When taking Laplace of everything:

$$\underbrace{Li' + Ri}_{\text{usual formulas}} + \frac{1}{C}\underbrace{\int_0^t i(y)\,dy}_{?!} = E$$

Recognize that $\int_0^t i(y)\,dy = (i * 1)$

usual formulas: $L(sI(s) - i(0)) + RI(s)$

$(i * 1)(t) = (1 * i)(t) = \int_0^t 1\cdot i(y)\,dy$
b/c if $f(x) = 1$ for all x, $f(t-y) = 1$

So we have $L\cdot s\cdot I(s) - L\cdot i(0) + R\cdot I(s) + \underbrace{\frac{1}{C}\mathcal{L}\{1 * i\}}_{\frac{1}{C}\mathcal{L}\{1\}\cdot\mathcal{L}\{i\} = \frac{1}{C}\frac{1}{s}\cdot I(s)} = \mathcal{L}\{E(t)\}$

=> Equation becomes:

$$L\cdot s\cdot I(s) - L\cdot i(0) + R\cdot I(s) + \frac{1}{Cs}I(s) = \mathcal{L}\{E(t)\}$$

Example: This is an example of a Volterra Integral Equation

(these appear in the statistical study of population dynamics, material science - esp. viscous materials - and in actuarial science - esp. in studying the risk of insolvency)

$$x(t) = 3\cos(t) + 5\underbrace{\int_0^t \sin(t-y)\,x(y)\,dy}_{(\sin * x)(t)}$$

Laplace of everything:

$$X(s) = 3\mathcal{L}\{\cos(t)\} + 5\mathcal{L}\{\sin * x\}$$

$$X(s) = \frac{3s}{s^2+1} + 5\mathcal{L}\{\sin(t)\}\cdot\mathcal{L}\{x(t)\}$$

$$X(s) = \frac{3s}{s^2+1} + \frac{5}{s^2+1}X(s)$$

$$X(s)\left(1 - \frac{5}{s^2+1}\right) = \frac{3s}{s^2+1}$$

$$X(s)\ \frac{s^2-4}{s^2+1} = \frac{3s}{s^2+1}$$

$$\boxed{X(s) = \frac{3s}{s^2-4}} \Rightarrow \boxed{x(t) = 3\cosh(2t)}$$