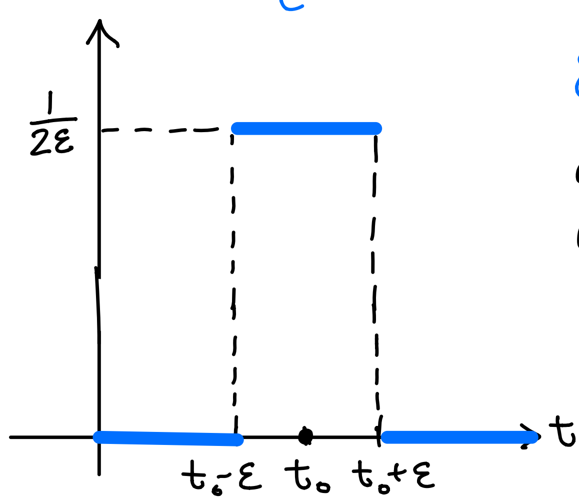
Dirac Delta Functions

Unit Impulse: Model the situation of an external force of large magnitude acting on a system (mechanical, electrical, oscillatory etc.) only for a very short time (a lightning strike; a golf ball given a sharp blow by a club etc.).

$$\delta_\varepsilon(t-t_0) = \begin{cases} 0, & \text{if } 0 \le t < t_0-\varepsilon \\ \frac{1}{2\varepsilon}, & \text{if } t_0-\varepsilon \le t < t_0+\varepsilon \\ 0, & \text{if } t \geq t_0+\varepsilon \end{cases} = \frac{1}{2\varepsilon}\mathbb{1}_{[t_0-\varepsilon,\, t_0+\varepsilon)}$$



$\delta_\varepsilon(t-t_0)$: An impulse of magnitude $\frac{1}{2\varepsilon}$ acting for (2ε) units of time, centered at $t=t_0$.

$$\int_0^\infty \delta_\varepsilon(t-t_0)\,dt = \frac{1}{2\varepsilon}\cdot 2\varepsilon = \boxed{1}$$

$$\Downarrow$$

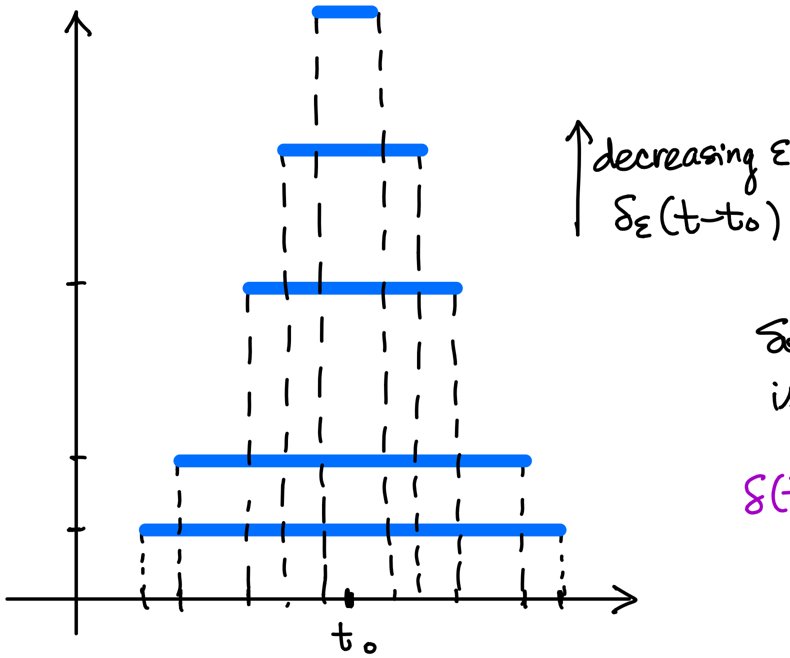
For all $\varepsilon>0$: $\boxed{\int_0^\infty \delta_\varepsilon(t-t_0)\,dt = 1}$

(why it's called "unit" impulse)

Dirac Delta Function: To model a strong external force acting <u>instantaneously</u> at $t=t_0$, we define:

$$\boxed{\delta(t-t_0) := \lim_{\varepsilon \to 0} \delta_\varepsilon(t-t_0)} \leftarrow \text{Dirac Delta Function}$$

Effectively, this lets the time interval $(t_0-\varepsilon, t_0+\varepsilon)$ around time t_0 decrease to 0. In the same time, the magnitude of the force, $\frac{1}{2\varepsilon}$, tends to ∞.



So, in the limit, $\delta(t-t_0)$ is the "impulse":

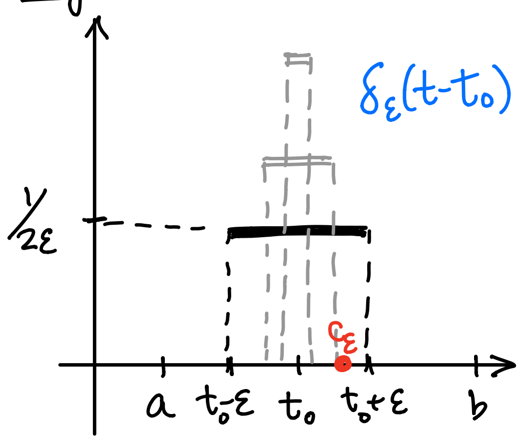
$$\delta(t-t_0) = \begin{cases} 0, & \text{if } t \neq t_0 \\ \infty, & \text{if } t = t_0 \end{cases}$$

As you can see, this is not really a "function". In fact, rigorously speaking, δ is a distribution (aka generalized function), an advanced concept which is the subject of its own branch of mathematics – distribution theory. This began with Laurent Schwartz trying to make sense of Dirac's strange "function".

Roughly speaking, a **distribution** is an operator acting on functions, defined by how it acts on functions. For the Dirac delta, **the defining action** is:

> For any continuous function f on an interval $[a,b]$ containing a point t_0:
>
> $$\boxed{\int_a^b f(t)\,\delta(t-t_0)\,dt = f(t_0)}$$

Proof:



$$\int_a^b f(t)\,\delta_\varepsilon(t-t_0)\,dt =$$

$$= \int_{t_0-\varepsilon}^{t_0+\varepsilon} f(t)\,\frac{1}{2\varepsilon}\,dt$$

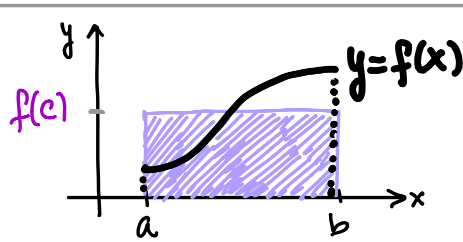
$$= \frac{1}{2\varepsilon}\left(\cancel{t_0}+\varepsilon-(\cancel{t_0}-\varepsilon)\right) f(c_\varepsilon)$$

$$= f(c_\varepsilon), \text{ for } \underline{\text{some }} c_\varepsilon \in [t_0-\varepsilon, t_0+\varepsilon].$$

Mean Value Thm. for Integrals:

If f is continuous on $[a,b]$, then there is at least one value $c \in [a,b]$ such that:

$$\int_a^b f(x)\,dx = (b-a)\,f(c)$$



$$\boxed{\int_a^b f(t)\,\delta_\varepsilon(t-t_0)\,dt = f(c_\varepsilon)}$$

Taking $\lim_{\varepsilon\to 0}$ above yields:

$$\underbrace{\lim_{\varepsilon\to 0}\int_a^b f(t)\,\delta_\varepsilon(t-t_0)\,dt}_{} = \underbrace{\lim_{\varepsilon\to 0} f(c_\varepsilon)}_{}$$

Glossing over why this is allowed, interchange limit & integral:

$$\int_a^b f(t)\underbrace{\lim_{\varepsilon\to 0}\delta_\varepsilon(t-t_0)}_{\delta(t-t_0)}dt$$

As $\varepsilon\to 0$, the interval $[t_0-\varepsilon, t_0+\varepsilon]$ collapses onto the point t_0:

$$\int_a^b f(t)\,\delta(t-t_0)\,dt = f(t_0) \qquad \blacksquare$$

Some important special cases of:

$$\boxed{\int_a^b f(t)\,\delta(t-t_0)\,dt = f(t_0)}$$

$a=0,\ b=\infty$

$$\boxed{\int_0^\infty f(t)\,\delta(t-t_0)\,dt = f(t_0)}$$

aka the "sifting property", because $\delta(t-t_0)$ "sifts" through all the values of f on $[0,\infty)$ to yield $f(t_0)$.

Taking $f(t)=e^{-st}$ above:

$$\underbrace{\int_0^\infty e^{-st}\,\delta(t-t_0)\,dt}_{\mathcal{L}\{\delta(t-t_0)\}} = e^{-st_0}$$

$$\boxed{\mathcal{L}\{\delta(t-t_0)\} = e^{-st_0}}$$

In particular, letting $t_0=0$ above:

$$\boxed{\mathcal{L}\{\delta(t)\} = 1}$$

$a=0,\ b=t_0$

$$\boxed{\int_0^{t_0} f(t)\,\delta(t_0-t)\,dt = f(t_0)}$$

This follows b/c δ is even, i.e.

$$\delta(t-t_0) = \delta(t_0-t)$$

But notice that the left side is really

$$(f*\delta)(t_0) = f(t_0)$$

In other words:

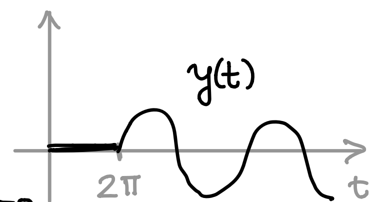
$$\boxed{f*\delta = f}$$

δ is the "identity" for the operation of convolution!

1. $y'' + y = 4\delta(t-2\pi)$ → models the motion of a mass on a spring that is given a sharp blow at $t = 2\pi$

$y(0)=0;\ y'(0)=0$ → mass begins at rest in the equilibrium position

$$\mathcal{L}\{y''+y\} = 4\mathcal{L}\{\delta(t-2\pi)\}$$



$$s^2Y(s) - sy(0) - y'(0) + Y(s) = 4e^{-2\pi s}$$

$$(s^2+1)Y(s) = 4e^{-2\pi s} \Rightarrow Y(s) = \frac{4e^{-2\pi s}}{s^2+1}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{4e^{-2\pi s}}{s^2+1}\right\} = 4\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}\Big|_{t\to t-2\pi} u_{2\pi}(t)$$

$$= 4\sin(t-2\pi)u_{2\pi}(t) = \boxed{4\sin(t)u_{2\pi}(t)}$$

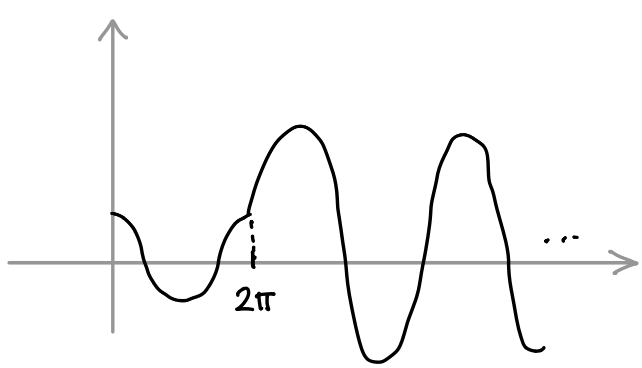
2. $y'' + y = 4\delta(t-2\pi)$

$y(0)=1;\ y'(0)=0$ → mass is released from rest 1 unit below the equilibrium point

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = 4e^{-2\pi s}$$

$$(s^2+1)Y(s) = s + 4e^{-2\pi s} \Rightarrow Y(s) = \frac{s}{s^2+1} + \frac{4e^{-2\pi s}}{s^2+1}$$

$$\Rightarrow \boxed{y(t) = \cos(t) + 4\sin(t)u_{2\pi}(t)} = \begin{cases} \cos(t), & \text{if } 0 \le t < 2\pi \\ \cos(t) + 4\sin(t), & \text{if } t \ge 2\pi \end{cases}$$



Remark: Since $t > 0$, the condition $y'(0) = 0$ should be interpreted as $\lim_{t\to 0^+} y'(t) = 0$.