Autonomous Equations:

An autonomous first order ODE is an equation of the form:

$$\boxed{\frac{dy}{dx} = f(y)} \qquad (*)$$

(No independent variable on the right side).

For example: $\dfrac{dy}{dx} = y^2 + \sin(y)$ (autonomous) $\qquad \dfrac{dy}{dx} = xy$ (not autonomous).

- An autonomous equation is <u>separable</u>: If $f(y) \neq 0$, we can write the equation as

$$\frac{1}{f(y)}\, dy = dx,$$

and solve by integrating both sides (if we can).

- What if $f(y) = 0$?
  - Any number $c \in \mathbb{R}$ such that $f(c) = 0$ is called a **critical point** (aka **equilibrium point** or **stationary point**) of the ODE $(*)$.
  - The constant functions $y \equiv c$, where c is any equilibrium pt. are the (only) constant solutions to the ODE $(*)$.

    To see this, let $y(x) = c$ for all x, where $f(c) = 0$.

    Then: $\dfrac{dy}{dx} = f(c) = 0$, so $y \equiv c$ is a solution to $(*)$.
  - The constant solutions $y \equiv c$ where c is an equilibrium pt. are called the **equilibrium solutions** (because they correspond to the situations where y does not vary as x increases or decreases.)

① $\dfrac{dy}{dx} = y-2$

- Equilibrium sol.: $\boxed{y=2}$
- To find the other solutions:

$$\int \frac{1}{y-2}\,dy = \int dx$$

$$\ln|y-2| + c = x$$

$$c(y-2) = e^x$$

$$y-2 = ce^x$$

$$\boxed{y = ce^x + 2}$$

② $\dfrac{dy}{dx} = (y-1)(y-2)$

- Equilibrium sol.: $\boxed{y=1}$ & $\boxed{y=2}$
- Other solutions:

$$\int \frac{1}{(y-1)(y-2)}\,dy = \int dx$$

$$\ln\left|\frac{y-2}{y-1}\right| = x + c$$

$$\boxed{\frac{y-2}{y-1} = ce^x} \text{ (implicit)}$$

$$y-2 = ce^x y - ce^x$$

$$(1-ce^x)\,y = 2 - ce^x$$

$$\boxed{y = \frac{2-ce^x}{1-ce^x}} \text{ (explicit)}$$

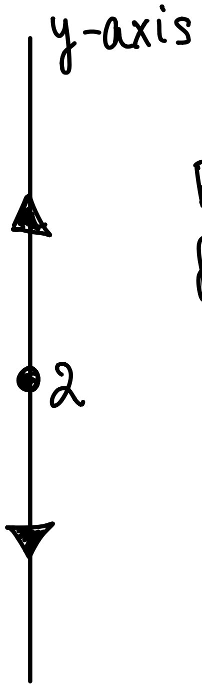
Phase Portraits: Look again at the autonomous equation

$$\frac{dy}{dx} = y-2$$

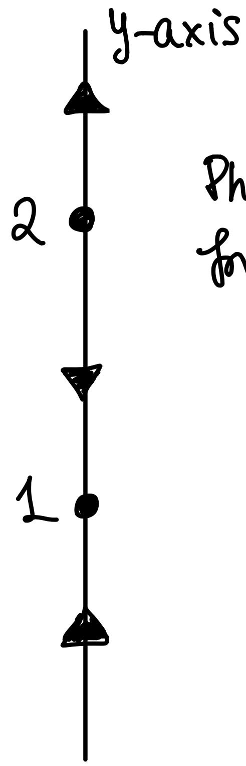
- If $y<2$, then $y' = y-2 < 0$, which tells us that any solution y will be decreasing in the region $y<2$.
- If $y>2$, then $y' = y-2 > 0$, so a solution y will be increasing when $y>2$.

y		1		2	
$y-1$	$-$	0	$+$		$+$
$y-2$	$-$		$-$	0	$+$
y'	$+$	0	$-$	0	$+$

Similarly, for $y' = (y-1)(y-2)$: any solution is increasing when $y \in (-\infty, 1) \cup (2, \infty)$ and decreasing for $y \in (1,2)$.



Phase portrait for $y' = y - 2$



Phase portrait for $y' = (y-1)(y-2)$

Generally: The phase portrait of an autonomous ODE

$$y' = f(y)$$

(assuming f, f' are continuous on some interval),

- Draw a vertical line representing the y-axis ("dependent variable" axis). This is called a phase line.
- Mark the equilibrium points on this line. The equilibrium (critical) points partition the phase line into distinct intervals.
- On each interval partitioned by critical points, any solution y will be either increasing or decreasing.

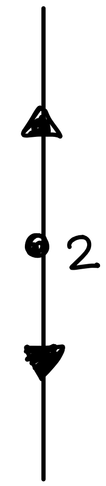
orient the line ↑   orient the line ↓

Why?

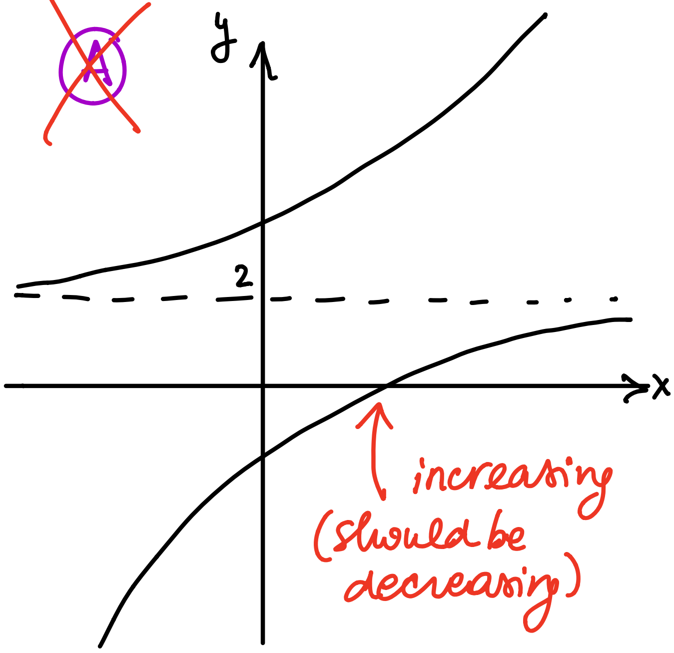
Recall that the equilibrium pts. are the zeros of the function f, and so f will be either ⊕ or ⊖ b/w critical pts. And, f is the derivative of y.

Look at $\frac{dy}{dx} = y - 2$ and its phase portrait.

Which of the following could be solutions to this equation?

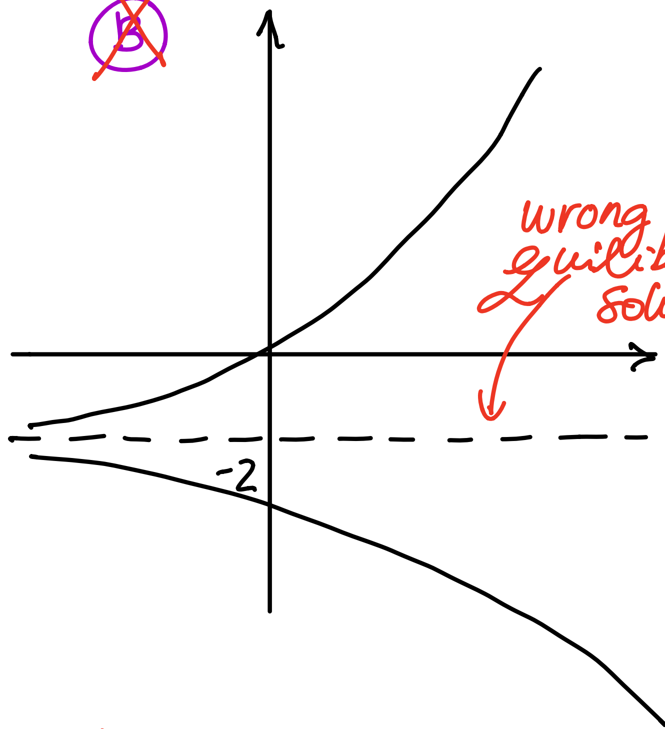


~~A~~



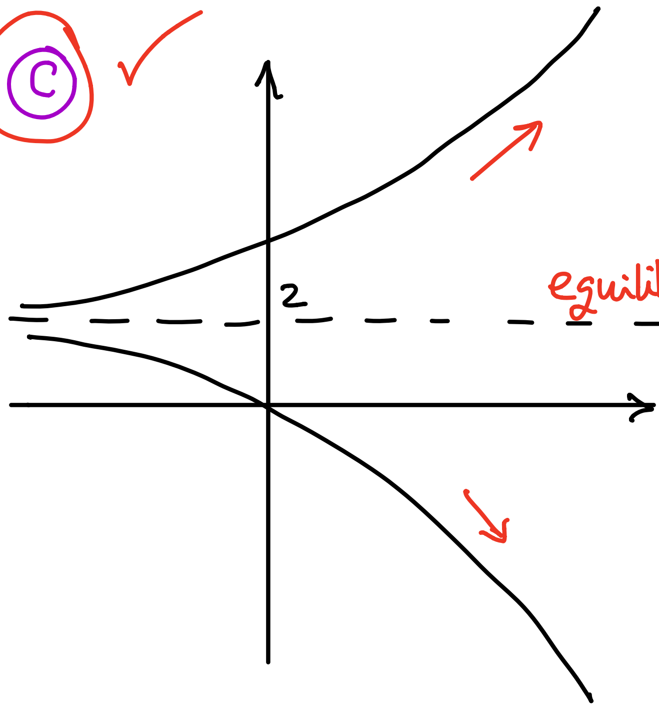
increasing (should be decreasing)

~~B~~



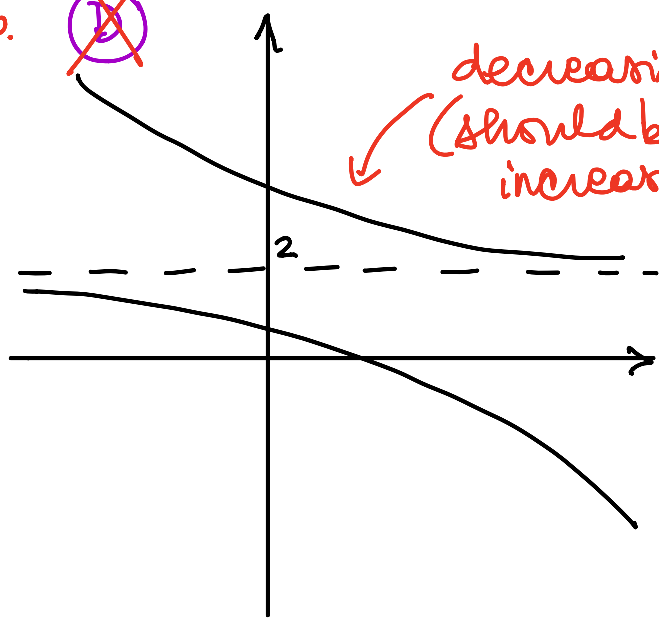
wrong equilibrium solution

C ✓



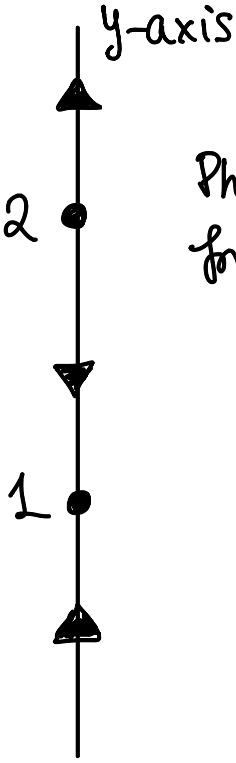
equilib.

~~D~~



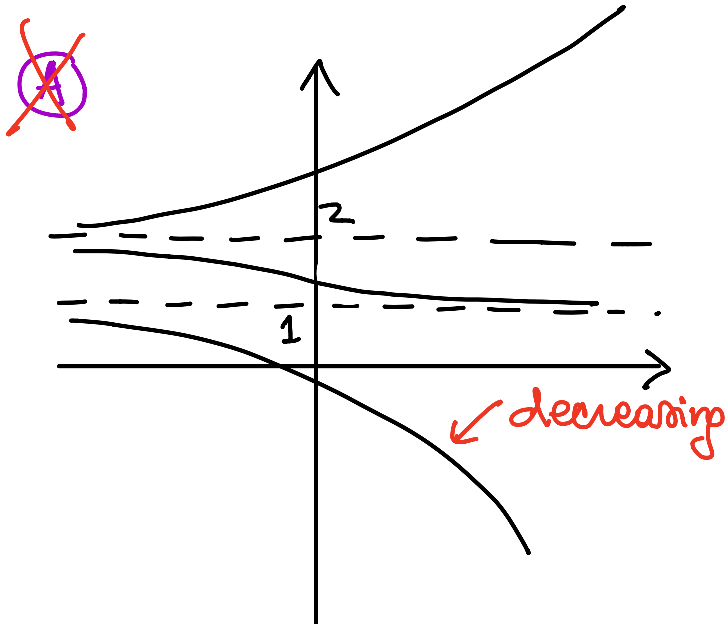
decreasing (should be increasing)

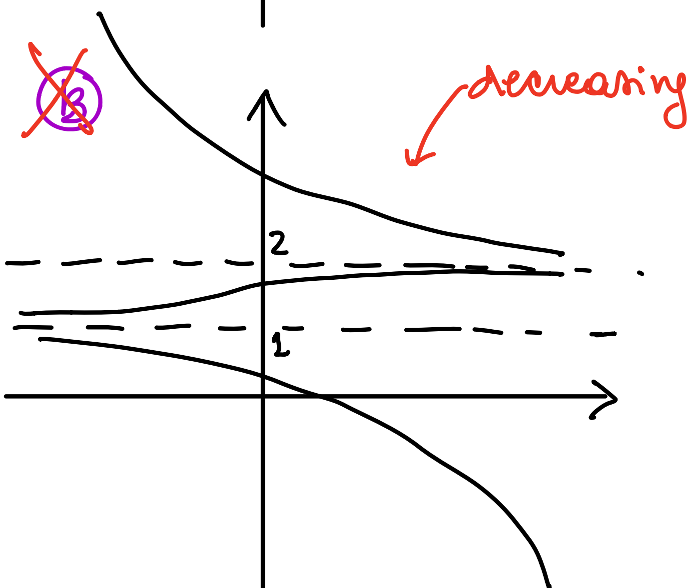
$$\frac{dy}{dx} = (y-1)(y-2)$$

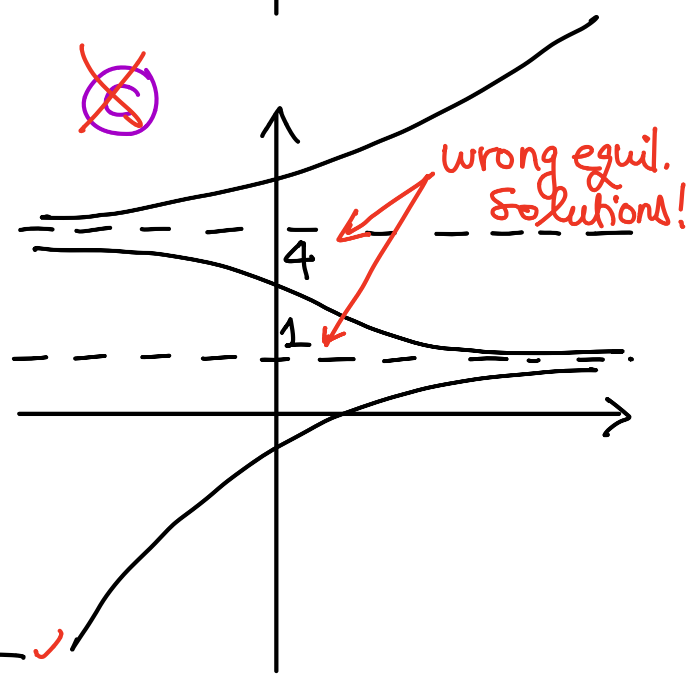


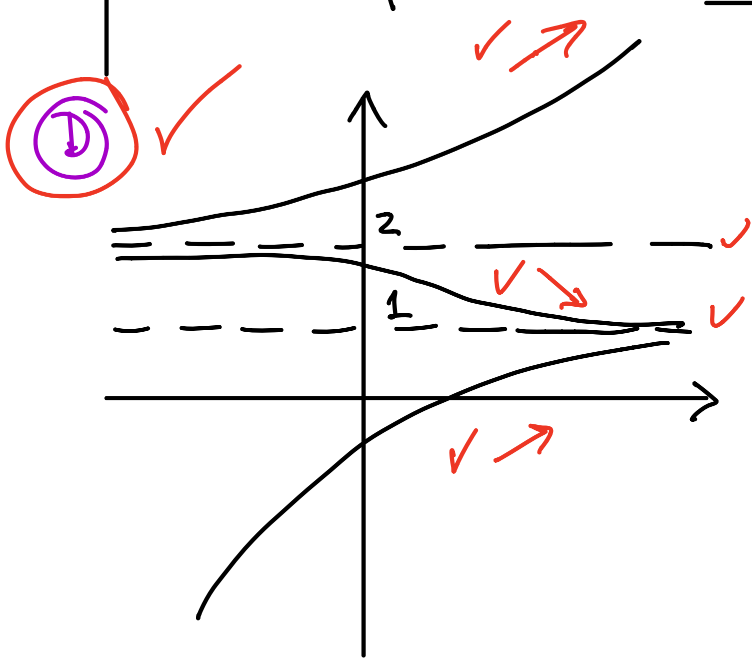
Phase portrait for $y' = (y-1)(y-2)$

Same question: which graphs below could belong to solutions of this equation?

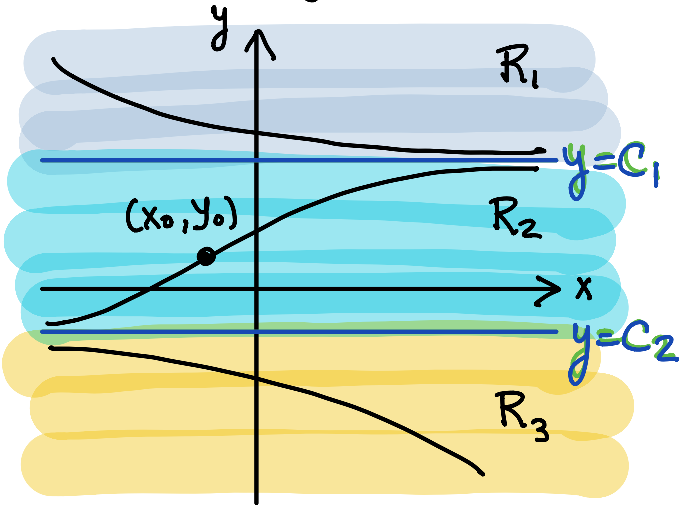








What can we learn from a phase portrait about the solutions to an autonomous ODE $y' = f(y)$? Suppose for simplicity that f and f' are both continuous functions of y. Then the hypotheses of the Existence & Uniqueness thm. are satisfied, so through any point (x_0, y_0) there passes exactly one solution curve.



The equilibrium solutions partition the plane into horizontal strips $R_1, R_2, .., R_k$. These strips reveal some important features of the solutions to the ODE:

① Solutions are "trapped" in these regions

If a point (x_0, y_0) belongs to a region R_i and $y(x)$ is the solution passing through that point, then $y(x)$ remains within R_i for all x (because two solution curves cannot intersect - uniqueness - and for $y(x)$ to leave the region, it would have to cross one of the equilibrium solutions).

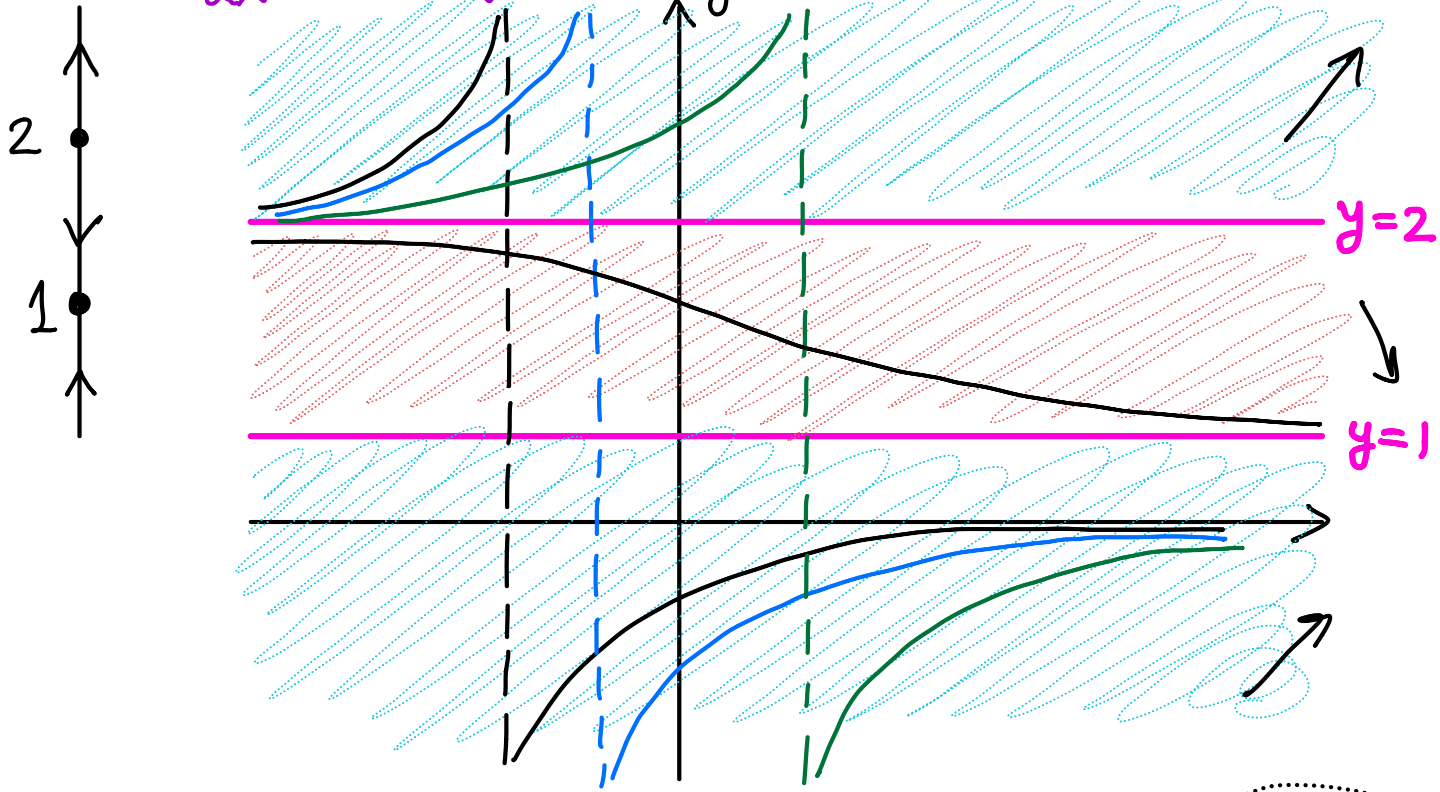
② Solutions are either increasing or decreasing in a region R_i

This is because $f(y)$, assumed continuous, can be either positive or negative b/w two of its zeros (the critical points).

③ The equilibrium solutions are horizontal asymptotes for the other solutions:

If $y(x)$ is a solution trapped in region R_i, and c is a critical point bounding the region (either from above or below) then $y(x)$ must approach c either as $x \to -\infty$ or $x \to \infty$.

Ex): $\frac{dy}{dx} = (y-1)(y-2)$



- All solutions in $[y>2]$ are increasing $\Rightarrow$ must have: $\lim\limits_{x\to-\infty} y(x) = 2$
- All solutions in $[1<y<2]$ are decreasing $\Rightarrow$ must have: $\lim\limits_{x\to-\infty} y(x) = 2$; $\lim\limits_{x\to\infty} y(x) = 1$.
- All solutions in $[y<2]$ are increasing $\Rightarrow$ must have $\lim\limits_{x\to\infty} y(x) = 1$.

Solutions to the ODE are: $\boxed{y = \frac{2-ce^x}{1-ce^x}}$ $\longrightarrow$ could have vertical asymptotes?

$1-ce^x = 0$; $\underbrace{\frac{1}{c} = e^x}_{\text{only possible when } c>0}$;

$\Rightarrow$ all solutions $y = \frac{ce^x-2}{ce^x-1}$ with $c>0$ have vertical asymptotes, at $x = \ln\left(\frac{1}{c}\right)$.

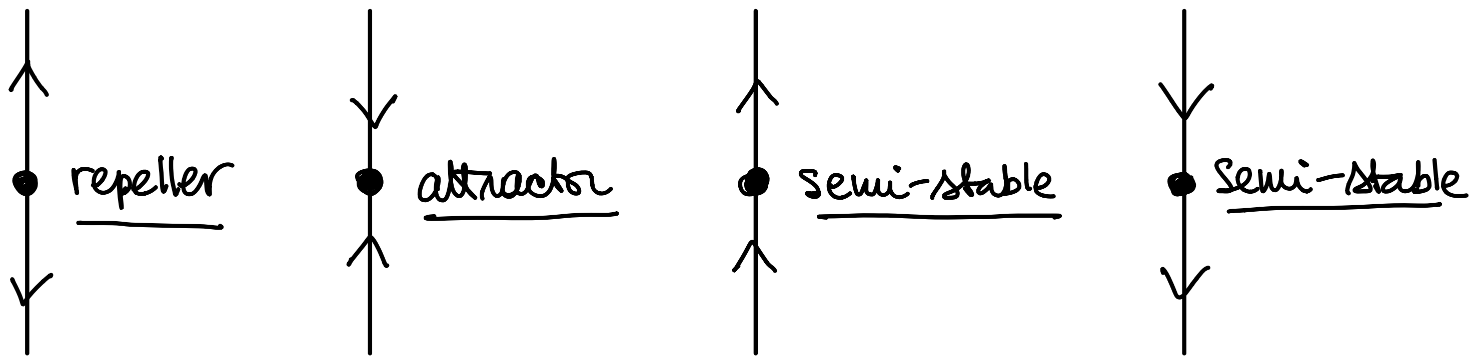
(These are the solutions in the blue regions)

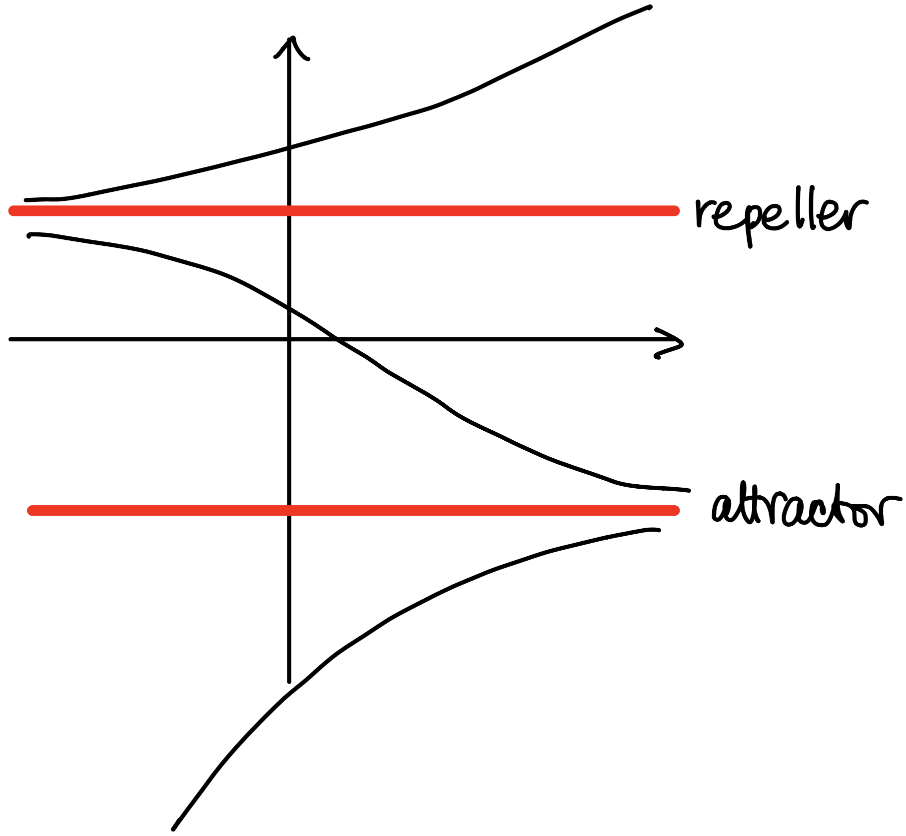
$\Rightarrow$ all solutions $y = \frac{ce^x-2}{ce^x-1}$ with $c<0$ have no vertical asymptotes

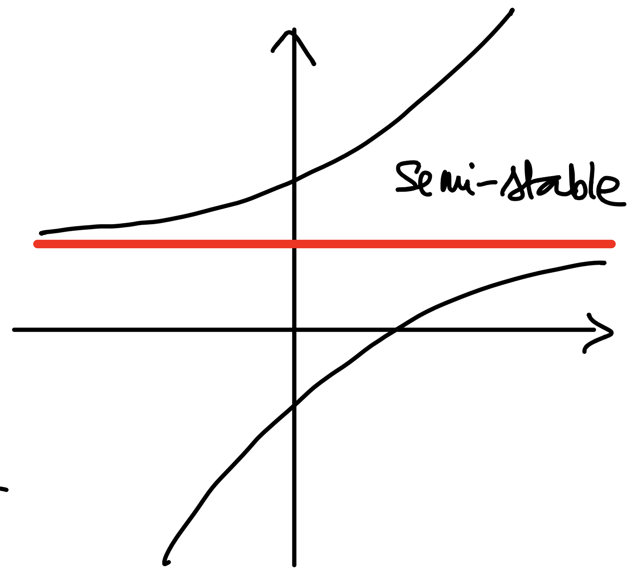
(These are in the middle purple region).

Attractors & Repellers

Four possible behaviors around a critical point in a phase portrait:

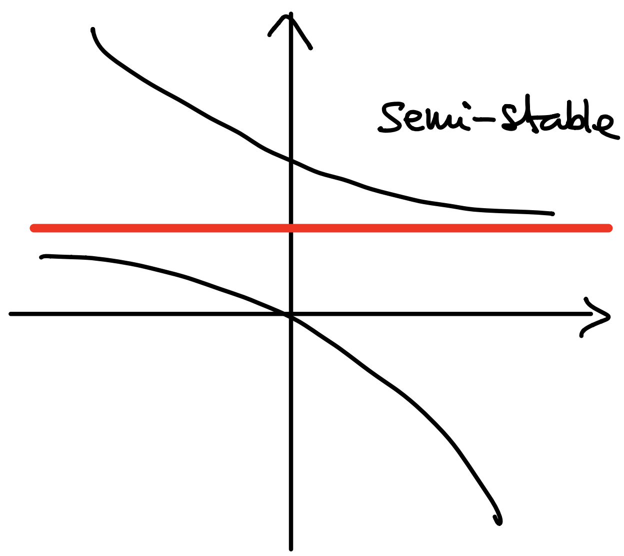






* **Repeller**: any solution passing close enough to $y=c$ moves away from c as $x \to \infty$

* **Attractor**: any solution passing close enough to $y=c$ moves towards c as $x \to \infty$.



* **Semi-stable**: behaves like an attractor on one side & like a repeller on the other side.

Population Models:

① Exponential Growth

Let $y(t)$ denote the population of a given species at time t.

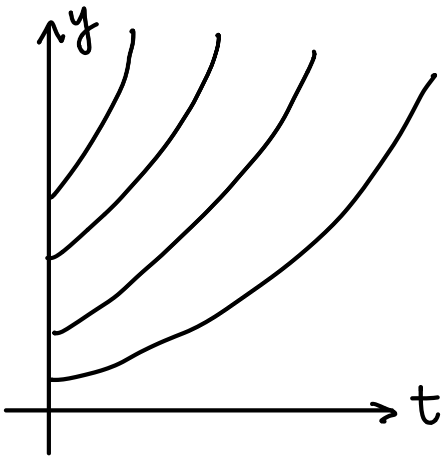
- Simplest hypothesis: the rate of change of y is proportional to the current value of y (if the population doubles, the number of births also doubles):

$$\boxed{\frac{dy}{dt} = ry}$$

where r is a non-zero constant called rate of growth (if $r>0$) / rate of decay (if $r<0$)

- Solve this subject to initial condition:

$$\boxed{y(0) = y_0}$$



$$\int \frac{1}{y}\,dy = \int r\,dt$$

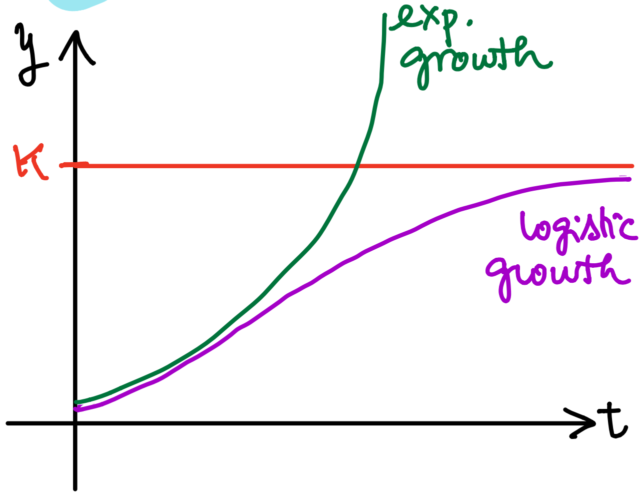
$$\left.\begin{aligned} \ln|y| &= rt + c \\ y &= ce^{rt} \\ t = 0,\ y = y_0 &\Rightarrow y_0 = c \end{aligned}\right\} \Rightarrow \boxed{y(t) = y_0 e^{rt}}$$

If $r>0$, this is exponential growth (& exponential decay if $r<0$).

* Reasonably accurate model for many populations, under ideal conditions.

* Problem: The ideal conditions cannot last forever (resources deplete). => this model is usually accurate only for limited periods of time.

② Logistic Growth



→ Essentially, this means that there is some number K, called the **carrying capacity** (of the environment) that bounds the population of the species.

$$\boxed{\frac{dy}{dt} = r\left(1 - \frac{1}{K}y\right)y}$$

Logistic Equation (aka **Verhulst equation**)

(K = maximum possible population).

* Remark: when the population is very small compared to K, the two models are basically indistinguishable. But, as the population grows:

~ the exponential model grows indefinitely

~ the logistic model approaches K and the growth rate approaches 0.

* The idea behind the model is that the relative growth rate $\frac{y'}{y}$ decreases as y approaches the carrying capacity K of the environment:

$$\frac{y'}{y} = r\underbrace{\left(1 - \frac{1}{K}y\right)}_{}$$

as $y \nearrow K$, this $\left(1-\frac{1}{K}y\right) \searrow 0$

(r = constant depending on environment)

Solve the logistic equation:

$$\boxed{\frac{dy}{dt} = r\left(1-\frac{1}{K}y\right)y}$$

→ autonomous equation

→ equilibrium solutions:

$y=0, \quad y=K.$

Separate variables: $\displaystyle\int \frac{1}{\left(1-\frac{1}{K}y\right)y}\,dy = \int r\,dt$

$$\frac{1}{\frac{K-y}{K}\,y} = \frac{K}{(K-y)y} = \frac{(K-y)+y}{(K-y)\cdot y}$$

or do partial fractions

$$= \frac{1}{y} + \frac{1}{K-y}$$

$$\Rightarrow \int\left(\frac{1}{y}+\frac{1}{K-y}\right)dy = \ln|y| - \ln|K-y|$$

$$= \ln\left|\frac{y}{K-y}\right|$$

$$\Rightarrow \ln\left|\frac{y}{K-y}\right| = rt + c$$

$$\frac{y}{K-y} = ce^{rt}$$

$$y = Kce^{rt} - ce^{rt}y$$

$$(1+ce^{rt})y = Kce^{rt}$$

$$y = \frac{Kce^{rt}}{1+ce^{rt}} = \frac{Kc}{c+e^{-rt}}$$

IVP: $y(0)=y_0$ (initial population)

$$\Rightarrow \quad y_0 = \frac{Kc}{c+1} \quad \Rightarrow c+1 = \frac{K}{y_0}c$$

$$\Rightarrow \left(\frac{K}{y_0}-1\right)c = 1 \quad \Rightarrow c = \frac{y_0}{K-y_0}$$

$$\Rightarrow y = \frac{K\cdot\frac{y_0}{K-y_0}}{\frac{y_0}{K-y_0}+e^{-rt}} = \frac{Ky_0}{y_0+(K-y_0)e^{-rt}}$$

$$\boxed{y(t) = \frac{Ky_0}{y_0+(K-y_0)e^{-rt}}}$$

→ Solution to the logistic equation w/ initial cond. $y(0)=y_0$.

Ex]: **3. A run-of-the-mill population problem:** Suppose a population of bacteria follows the logistic growth model. Suppose further that the initial population is 3mg of bacteria, the carrying capacity is 100mg, and the growth parameter is $r = 0.2$/hour.

y_0 $\quad$ K

a). At what time does the population reach 20mg?

b). At what time does the population reach 200mg?

$y_0 = 3$ (mg)
$K = 100$ (mg)
$r = 0.2$ /hour

$$\frac{dy}{dt} = 0.2\left(1 - \frac{1}{100}y\right)y$$

$$y(t) = \frac{100 \cdot 3}{3 + (100-3)e^{-0.2t}} = \frac{300}{3 + 97e^{-0.2t}}$$

a). $y(t) = 20 \Rightarrow \frac{300}{3+97e^{-0.2t}} = 20$

$$3 + 97e^{-0.2t} = 15$$
$$e^{-0.2t} = \frac{12}{97}$$
$$-0.2t = \ln(12/97)$$
$$\boxed{t = \frac{\ln(97/12)}{0.2}} \approx 10.45 \text{ hours}$$

b). $y(t) = 200$

No work needed: never, b/c 200 exceeds the carrying capacity (100)!

Work in case you don't see this:

$$\frac{300}{3+97e^{-0.2t}} = 200 \Rightarrow 3 + 97e^{-0.2t} = \frac{3}{2}$$

$$97e^{-0.2t} = -\frac{3}{2}$$

not possible ($e^{\text{anything}} > 0$).