2.2 | Separable ODE's

Def.: A separable ODE is one which can be written as:

$$\boxed{\frac{dy}{dx} = \frac{g(x)}{h(y)}}$$

Note: They aren't always given in this form.

Solution: Write as

$$h(y)dy = g(x)dx$$

Integrate both sides.

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(1) $\frac{dy}{dx} = -\frac{x}{y}$

$$y\,dy = -x\,dx$$

$$\int y\,dy = -\int x\,dx$$

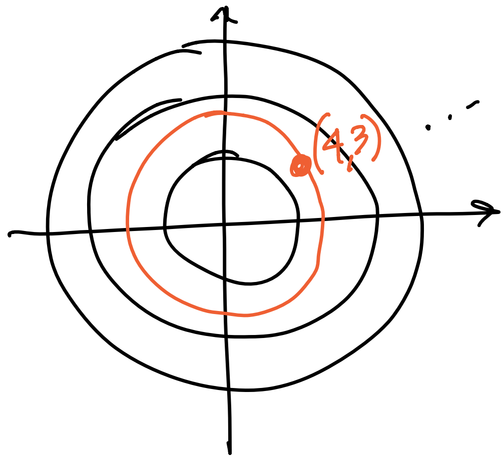
$$\frac{y^2}{2} = -\frac{x^2}{2} + C$$

$$\frac{x^2+y^2}{2} = C$$

$x^2+y^2 = 2C$ (Relabel as "c" → c is generic)

$\boxed{x^2+y^2=c}$ → General Solution
→ Implicit Solution (that's ok!)

What do the solutions look like? Concentric circles centered at the origin ⟼ "Integral Curves"



IVP: $y(4)=3$

$$x^2+y^2=C$$

$x=4 \quad y=3$

$$4^2+3^2=C \Rightarrow C=25$$

$$\boxed{x^2+y^2=25}$$

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② $xe^{-y}\sin x\,dx - y\,dy = 0$

$$xe^{-y}\sin x\,dx = y\,dy \qquad | \cdot e^y$$

$$\int x\sin x\,dx = \int ye^y\,dy$$

$$\boxed{-x\cos x+\sin x = ye^y - e^y + C}$$

Integration Details:

$\int x\sin x\,dx$ $\qquad (\int u\,dv = uv - \int v\,du)$

$u=x; \quad dv=\sin x$

$du=dx; \quad v=-\cos x$

$$-x\cos x+\int \cos x\,dx = -x\cos x+\sin x+C$$

$\int ye^y\,dy$

$u=y; \quad dv=e^y$

$du=dy; \quad v=e^y$

$$ye^y-\int e^y\,dy = ye^y-e^y+C.$$

③ $(1+x)\,dy - y\,dx = 0$ $\qquad \dfrac{dy}{dx} = \dfrac{y}{1+x}$

$$(1+x)\,dy = y\,dx$$

$$\int \frac{1}{y}\,dy = \int \frac{1}{1+x}\,dx$$

When dividing by y, we are assuming $y \not\equiv 0$

$$\ln|y| = \ln|1+x| + C$$

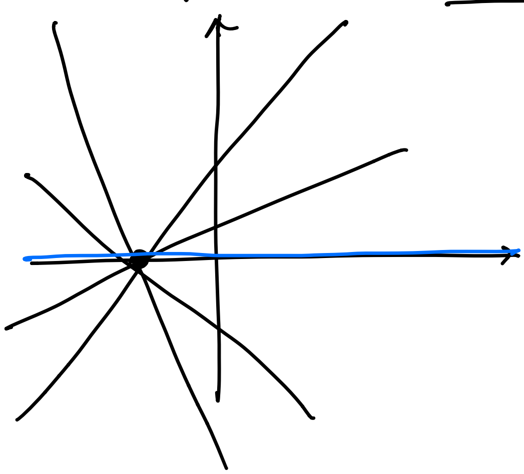
$$|y| = e^{\ln|1+x| + C}$$

$$= |1+x|e^{C}$$

$$= \underbrace{\pm e^{C}}_{\text{relabel as } c}(1+x)$$

$\Rightarrow \boxed{y = c(1+x)}$

Integral Curves? Lines passing thru $(-1,0)$



What if $y \equiv 0$? (Meaning that y is the identically 0 function: $y(x) = 0$, for all x)

Is $y=0$ a solution?

i.e. does it satisfy $\dfrac{dy}{dx} = \dfrac{y}{1+x}$?

Yes.

So $y=0$ is a solution. Is it represented in $y = c(1+x)$?

Yes – take $c=0$!

So: $\boxed{y = c(1+x)}$ full solution.