The Gamma Function

- The Gamma function is defined for all $x>0$ by the integral:

$$\boxed{\Gamma(x) := \int_0^\infty e^{-r} r^{x-1}\, dr}$$

- This function serves as a continuous analogue to the factorial:

$$\Gamma(x+1) = \int_0^\infty e^{-r} r^{x}\, dr = \int_0^\infty (-e^{-r})' r^x\, dr$$

$x>0 \Rightarrow x+1>1$

$$= \underbrace{-e^{-r} r^x \Big|_{r=0}^{\infty}}_{0} + \int_0^\infty e^{-r} \cdot x r^{x-1}\, dr$$

$$= x \underbrace{\int_0^\infty e^{-r} r^{x-1}\, dr}_{\Gamma(x)} \Rightarrow \boxed{\Gamma(x+1) = x\,\Gamma(x)} \quad (*)$$

- Compute $\Gamma(1)$:

$$\Gamma(1) = \int_0^\infty e^{-r}\, dr = -e^{-r}\Big|_0^\infty = 1 \Rightarrow \Gamma(1) = 1$$

Using $(*)$:

$$\Gamma(2) = 1\cdot\Gamma(1) = 1 = 1!$$

$$\Gamma(3) = 2\cdot\Gamma(2) = 2\cdot 1 = 2!$$

$$\Gamma(4) = 3\cdot\Gamma(3) = 3\cdot 2\cdot 1 = 3!$$

$$\Gamma(5) = 4\cdot\Gamma(4) = 4\cdot 3\cdot 2\cdot 1 = 4!$$

$$\vdots$$

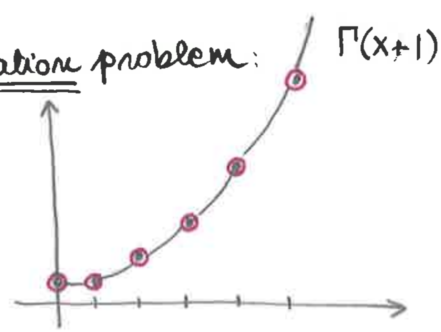
$$\boxed{\Gamma(n) = (n-1)!} \quad \text{for all positive integers } n$$

- Note that for $n=1$: $\Gamma(1) = 0!$ and we computed $\Gamma(1) = 1$.

  $\Rightarrow$ reason why $0! = 1$.

- $\Gamma(x)$ can be thought of as a solution to the interpolation problem:

  Suppose we are given the set of points in the plane: $(n, n!)$ for all integers $n = 0, 1, 2, \dots$ Is there a continuous function that "joins" all of these points? Yes: $f(x) = \Gamma(x+1)$.



The Gamma Function and the Laplace Transform:

Let us try to compute the Laplace transform of a general power function, i.e. $f(t) = t^p,\ p \geq 0$:

$$\mathcal{L}\{t^p\} = \int_0^\infty e^{-st} t^p dt$$

Change of variable: $r = st$, $dr = s\,dt$

$$= \int_0^\infty e^{-r} \left(\frac{r}{s}\right)^p \frac{1}{s} dr$$

$$= \frac{1}{s^{p+1}} \underbrace{\int_0^\infty e^{-r} r^p dr}_{\Gamma(p+1)} \Rightarrow \boxed{\mathcal{L}\{t^p\} = \frac{\Gamma(p+1)}{s^{p+1}}} \quad (\forall)\, p > -1$$

- If p is a positive integer $p = n$: $\mathcal{L}\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}} = \frac{n!}{s^{n+1}}$ (the formula we already know).

$$\boxed{\Gamma(1/2) = \sqrt{\pi}}$$

$$\Gamma(1/2) = \int_0^\infty e^{-r} r^{-1/2} dr$$

$$= \int_0^\infty e^{-r} \frac{1}{\sqrt{r}} dr$$

Change of variable: $x = \sqrt{r}$ $\Rightarrow dx = \frac{1}{2\sqrt{r}} dr$

$$= \int_0^\infty e^{-x^2} 2dx$$

$$= 2\int_0^\infty e^{-x^2} dx$$

Gaussian Integral: $\boxed{\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}}$

$$= 2 \cdot \frac{\sqrt{\pi}}{2} = \boxed{\sqrt{\pi}}$$

$$\Rightarrow \boxed{\mathcal{L}\{t^{-1/2}\} = \frac{\sqrt{\pi}}{s^{1/2}}}$$

$$\Rightarrow \boxed{\mathcal{L}\{t^{1/2}\} = \frac{\sqrt{\pi}}{2s^{3/2}}}$$

$$\mathcal{L}\{t^{-1/2}\} = \frac{\Gamma(1/2)}{s^{1/2}} = \frac{\sqrt{\pi}}{s^{1/2}}$$

$$\frac{\sqrt{\pi}}{s^{1/2}} = \mathcal{L}\{t^{-1/2}\} = \mathcal{L}\left\{\frac{1}{\sqrt{t}}\right\} = 2\mathcal{L}\{y'(t)\} \quad \text{where } y(t) = \sqrt{t}$$

$$= 2(sY(s) - y(0))$$

$$= 2s\,\mathcal{L}\{\sqrt{t}\}$$

$$\Rightarrow \mathcal{L}\{\sqrt{t}\} = \frac{1}{2s}\frac{\sqrt{\pi}}{\sqrt{s}} = \frac{\sqrt{\pi}}{2s^{3/2}}$$

Another way to show $\mathcal{L}\{\sqrt{t}\} = \dfrac{\sqrt{\pi}}{2s^{3/2}}$:

$$\mathcal{L}\{t^{1/2}\} = \frac{\Gamma(3/2)}{s^{3/2}}$$

How to find $\Gamma(3/2)$?

$$\Gamma(3/2) = \Gamma(1+1/2) = \frac{1}{2}\Gamma(1/2) = \frac{1}{2}\sqrt{\pi} \implies \mathcal{L}\{t^{1/2}\} = \frac{\sqrt{\pi}}{2s^{3/2}}$$