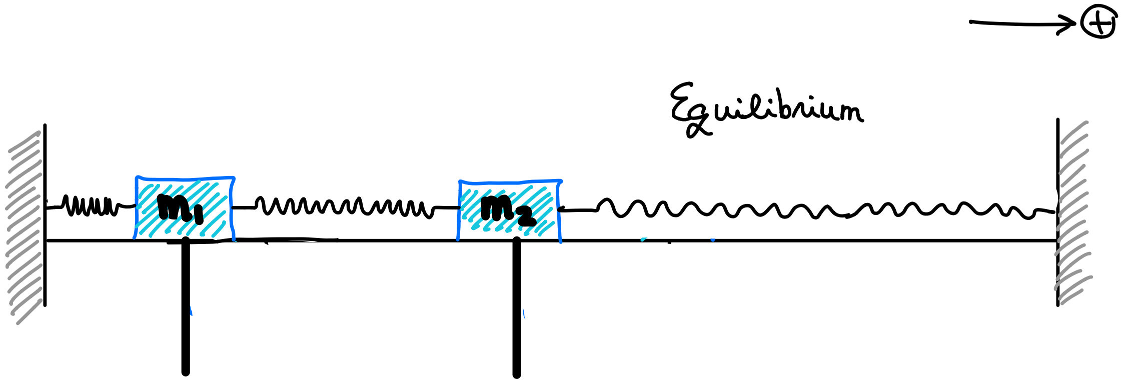
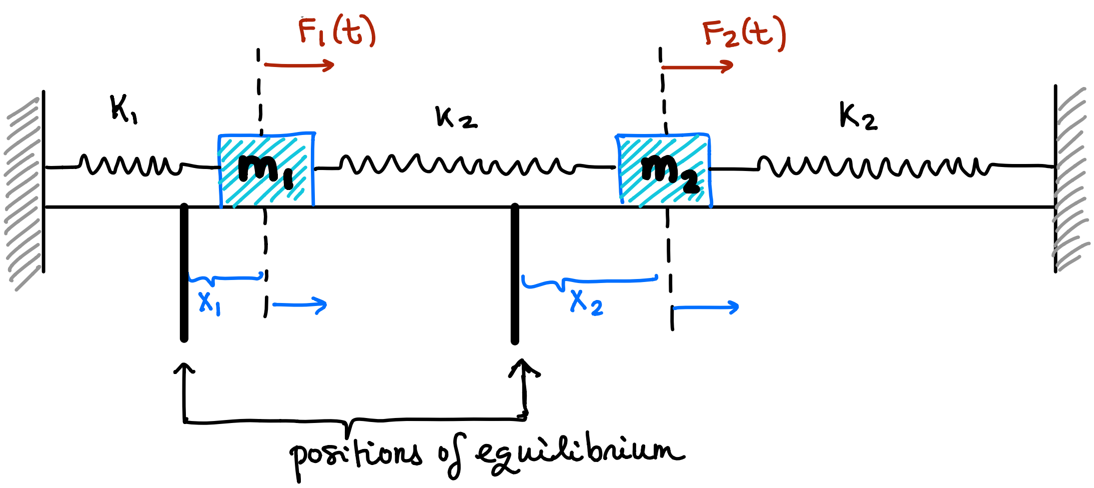
Systems of First-Order Linear Eqns.: Motivation

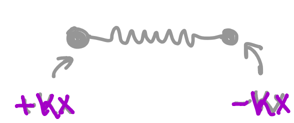




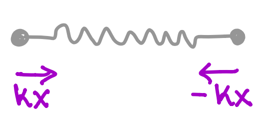
Two masses m_1, m_2 constrained by 3 springs w/ spring constants K_1, K_2, K_3, acted upon by external forces $F_1(t)$, $F_2(t)$. (Ignore friction)

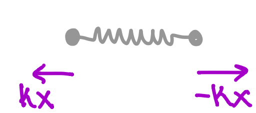
Coordinates x_1, x_2 of the 2 masses?

Remark: A spring w/ constant K, w/ masses attached at both ends, extended x:



$x > 0$ (Spring is stretched): $x < 0$ (Spring is compressed):



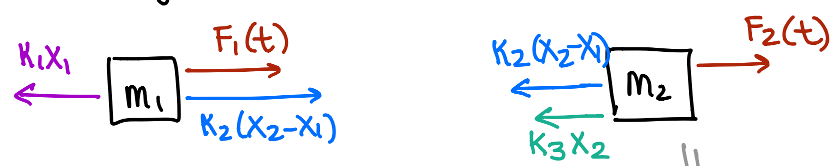


In our picture: $x_2 > x_1 > 0$, so:

Spring 1: stretched (x_1)

Spring 2: stretched $(x_2 - x_1) > 0$

Spring 3: compressed $(-x_2)$



$$m_1 x_1'' = F_1 + k_2(x_2 - x_1) - k_1 x_1$$
$$= F_1 + k_2 x_2 - k_2 x_1 - k_1 x_1$$

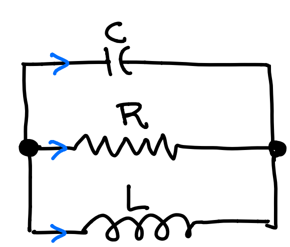
$$m_2 x_2'' = F_2 - k_2(x_2 - x_1) - k_3 x_2$$

System of differential equations!

$$\begin{cases} m_1 x_1''(t) = -(k_1 + k_2)\, x_1(t) + k_2\, x_2(t) + F_1(t) \\ m_2 x_2''(t) = k_2\, x_1(t) - (k_2 + k_3)\, x_2(t) + F_2(t) \end{cases}$$

Another practical example: a parallel LRC circuit:
the voltage drop V across the capacitor, and the current I through the inductor are described by a system:

$$\begin{cases} \dfrac{dI}{dt} = \dfrac{V(t)}{L} \\ \dfrac{dV}{dt} = -\dfrac{I(t)}{C} - \dfrac{V(t)}{RC} \end{cases}$$



Our main focus will be on systems of first-order equations. These are extremely important for a few reasons:

1. Equations of higher order can always be expressed as such a system. (below)

2. Numerical approaches: almost all codes for generating numerical approximations to solutions of ODE's are written for systems of first-order equations

Example: $\boxed{u''(t) + \frac{1}{2}u'(t) + u(t) = 0}$ → Second order eqn.

Let $x_1(t) := u(t)$ and $x_2(t) := u'(t)$

$\Rightarrow x_1'(t) = u'(t) = x_2(t)$

$\Rightarrow x_2'(t) = u''(t)$

So we can replace u, u', u'' in the original ODE by:

$$x_2'(t) + \frac{1}{2}x_2(t) + x_1(t) = 0$$

$$\Rightarrow \begin{cases} x_1'(t) = x_2(t) \\ x_2'(t) = -x_1(t) - \frac{1}{2}x_2(t) \end{cases}$$ System of 1st order ODEs

Solving Systems of ODEs w/ the Laplace transform:

① $\begin{cases} \dfrac{dx}{dt} = x + y \\ \dfrac{dy}{dt} = 2x \end{cases}$

$x(0) = 1; \; y(0) = 0.$

Same as before: Laplace everything:

$$\begin{cases} sX(s) - \underbrace{x(0)}^{1} = X(s) + Y(s) \\ sY(s) - \underbrace{y(0)}_{0} = 2X(s) \end{cases}$$

$$\begin{cases} sX(s) - X(s) - Y(s) = 1 \\ -2X(s) + sY(s) \quad = 0 \end{cases}$$

$$\begin{cases} (s-1)X(s) - Y(s) = 1 & \Big| \cdot 2 \\ -2X(s) + sY(s) = 0 & \Big| \cdot (s-1) \end{cases}$$

$$2(s-1)X(s) - 2Y(s) = 2$$
$$-2(s-1)X(s) + s(s-1)Y(s) = 0$$

$\oplus$ $\quad / \quad$ $[\underbrace{s(s-1)-2}_{s^2-s-2 = (s-2)(s+1)}]\,Y(s) = 2$

$$\boxed{Y(s) = \frac{2}{(s-2)(s+1)}}$$

$$X(s) = \frac{s}{2}Y(s) \Rightarrow \boxed{X(s) = \frac{s}{(s-2)(s+1)}}$$

$$x(t) = \mathcal{L}^{-1}\left\{\frac{s}{(s-2)(s+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{2/3}{s-2}\right\} + \mathcal{L}^{-1}\left\{\frac{1/3}{s+1}\right\}$$
$$= \frac{2}{3}e^{2t} + \frac{1}{3}e^{-t}$$

$$\frac{s}{(s-2)(s+1)} = \frac{A}{s-2} + \frac{B}{s+1}$$

$$s = A(s+1) + B(s-2)$$

$s = -1: \quad -1 = -3B \Rightarrow B = 1/3$

$s = 2: \quad 2 = 3A \Rightarrow A = 2/3$

$$y(t) = \mathcal{L}^{-1}\left\{ \frac{2}{(s-2)(s+1)} \right\} = \mathcal{L}^{-1}\left\{ \frac{2/3}{s-2} - \frac{2/3}{s+1} \right\}$$

$$\frac{(s+1)-(s-2) = 3}{(s+1)(s-2)}$$

$$= \frac{2}{3}e^{2t} - \frac{2}{3}e^{-t}$$

$$\Rightarrow \text{Solution: } \begin{cases} x(t) = \frac{2}{3}e^{2t} + \frac{1}{3}e^{-t} \\ y(t) = \frac{2}{3}e^{2t} - \frac{2}{3}e^{-t} \end{cases}$$

Verify?

$$x' = \frac{4}{3}e^{2t} - \frac{1}{3}e^{-t}$$

$$y' = \frac{4}{3}e^{2t} + \frac{2}{3}e^{-t}$$

$$\begin{cases} x' = x + y \\ y' = 2x \end{cases} \; ?$$

$$x + y = \frac{4}{3}e^{2t} - \frac{1}{3}e^{-t} = x' \quad \checkmark$$

$$2x = \frac{4}{3}e^{2t} + \frac{2}{3}e^{-t} = y' \quad \checkmark$$