Eigenvalues & Eigenvectors

$A\vec{x} = \vec{y}$ → linear transformation that "transforms" the vector $\vec{x}$ into the vector $\vec{y}$.

Vectors that are transformed into multiples of themselves under A are very important (in general, but also for ODEs)

Def.: For an $n \times n$ matrix A, if for some scalar λ and vector $\vec{v}$:

$$\boxed{A\vec{v} = \lambda\vec{v}}$$

then λ is called an eigenvalue of A, w/ corresponding eigenvector $\vec{v}$.

$$A\vec{v} - \lambda\vec{v} = 0 \Rightarrow (A - \lambda I)\vec{v} = 0$$

where I = identity matrix

$$I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

This is a homogeneous linear system, and the only situation in which it has non-trivial solutions is when $\boxed{\det(A - \lambda I) = 0}$ → the "characteristic equation" of the matrix A.

Example: $A = \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}$

$$A - \lambda I = \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

Char. Eqn.?

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 2 \\ 1 & 1-\lambda \end{vmatrix} = -\lambda(1-\lambda) - 2$$
$$= -\lambda + \lambda^2 - 2$$
$$= \lambda^2 - \lambda - 2$$
$$= (\lambda - 2)(\lambda + 1)$$

⇒ Eigenvalues: $\lambda_1 = 2$ $\lambda_2 = -1$

Eigenvector(s) for $\lambda_1 = 2$? Solve the linear system

$$(A - 2I)\vec{v} = 0 \qquad \text{(Solve for } \vec{v}\text{)}$$

$$A - 2I = \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} -2 & 2 & 0 \\ 1 & -1 & 0 \end{array}\right) \xrightarrow[R_1 : (-2)]{} \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & -1 & 0 \end{array}\right) \xrightarrow[R_1 - R_2]{} \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

Second eqn. gives no new information!

=> only information we have is $\boxed{v_1 - v_2 = 0}$ => $\boxed{v_1 = v_2}$

(2 unknowns, 1 equation – this is expected).

=> Any vector of the form $\vec{v} = \begin{pmatrix} c \\ c \end{pmatrix} = c \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

is an eigenvector corresponding to the eigenvalue $\lambda_1 = 2$.

Check:

$$A \begin{pmatrix} c \\ c \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c \\ c \end{pmatrix} = \begin{pmatrix} 2c \\ c + c \end{pmatrix} = 2 \cdot \begin{pmatrix} c \\ c \end{pmatrix} \quad \checkmark$$

We usually choose a "nice" representative, say

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and say $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is "the" eigenvector corresponding to λ_1, with the understanding that any scalar multiple of $\vec{v}_1$ is also an eigenvector for λ_1.

Homogeneous Linear Systems of ODE's w/ Constant Coefficients

$$\boxed{\vec{x}' = A\vec{x}}$$

- $\vec{x}$ is the vector of unknown functions: $\vec{x} = \vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$
- $\vec{x}'$ denotes the vector of 1^{st} derivatives:

$$\vec{x}' = \vec{x}'(t) = \begin{pmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{pmatrix}$$

- A is an $n \times n$ matrix of real numbers (the coefficients).

Ex: $\begin{cases} \dfrac{dx_1}{dt} = 2x_1(t) + 3x_2(t) \\ \dfrac{dx_2}{dt} = 2x_1(t) + x_2(t) \end{cases}$

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad A = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}$$

Look for a solution to $\vec{x}' = A\vec{x}$ of the form:

$$\vec{x} = \vec{v} e^{\lambda t} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} e^{\lambda t}$$

where $\vec{v}$ is a vector of constants.

$\Rightarrow \boxed{\vec{x}' = \vec{v}\lambda e^{\lambda t}} \Rightarrow$ $A\vec{x} - \vec{x}' = \vec{0}$ becomes

$$A\vec{v}e^{\lambda t} - \lambda\vec{v}e^{\lambda t} = \vec{0}$$

$$\boxed{(A\vec{v} - \lambda\vec{v})e^{\lambda t} = \vec{0}}$$

$$\Updownarrow$$

$$\boxed{A\vec{v} = \lambda\vec{v}}$$

$\vec{v}$ is an eigenvalue of A corresponding to the eigenvalue λ!

When an $n \times n$ matrix has n distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ then a set of n linearly indp. eigenvectors $\vec{v}_1, \ldots, \vec{v}_n$ can always be found. In this case,

$$\vec{x}_1 = e^{\lambda_1 t}\vec{v}_1,\ \vec{x}_2 = e^{\lambda_2 t}\vec{v}_2, \ldots,\ \vec{x}_n = e^{\lambda_n t}\vec{v}_n$$

are n linearly indp. solutions to the system.

$$\Downarrow$$

General Solution to Homogeneous Systems: $\vec{x}' = A\vec{x}$

Case 1: A has n distinct real eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ w/ corresponding eigenvectors $\vec{v}_1, \ldots, \vec{v}_n$. Then the general solution to the system is:

$$\vec{x} = c_1\vec{v}_1 e^{\lambda_1 t} + c_2\vec{v}_2 e^{\lambda_2 t} + \ldots + c_n\vec{v}_n e^{\lambda_n t}$$

Example: $\vec{x}' = A\vec{x}$, with $A = \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}$

We already saw: eigenvalues $\lambda_1 = 2$, $\lambda_2 = -1$.

We found the eigenvector for $\lambda_1 = 2$: $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Similarly, for $\lambda_2 = -1$:

Solve $(A + I)\vec{v} = \vec{0}$ $\quad \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 1 & 2 & 0 \end{array}\right) \xrightarrow[R_2 - R_1]{} \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$$v_1 + 2v_2 = 0 \Rightarrow v_1 = -2v_2 \Rightarrow \vec{v} = \begin{pmatrix} -2c \\ c \end{pmatrix} = c\begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$\vec{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

Check?

$$\begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2+1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} = (-1)\cdot\begin{pmatrix} -2 \\ 1 \end{pmatrix}. \quad \checkmark$$

$\Rightarrow$ General Solution to

$$\begin{cases} x_1' = 2x_2 \\ x_2' = x_1 + x_2 \end{cases} \qquad \vec{x}' = A\vec{x}$$

is:

$$\boxed{\vec{x} = c_1 e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} -2 \\ 1 \end{pmatrix}}$$

$$= \begin{pmatrix} c_1 e^{2t} - 2c_2 e^{-t} \\ c_1 e^{2t} + c_2 e^{-t} \end{pmatrix} = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \vec{x}$$

Check?

$$\Rightarrow \vec{x}' = \begin{pmatrix} 2c_1 e^{2t} + 2c_2 e^{-t} \\ 2c_1 e^{2t} - c_2 e^{-t} \end{pmatrix} \overset{?}{=} A\vec{x} = \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} c_1 e^{2t} - 2c_2 e^{-t} \\ c_1 e^{2t} + c_2 e^{-t} \end{pmatrix}$$

yes

$$= \begin{pmatrix} 2c_1e^{2t} + 2c_2e^{-t} \\ 2c_1e^{2t} - c_2e^{-t} \end{pmatrix}$$

General Solution to Homogeneous Systems: $\vec{x}' = A\vec{x}$

Case 2: Complex Eigenvalues : If

$$\lambda = \alpha + i\beta \quad \text{and} \quad \bar{\lambda} = \alpha - i\beta$$

are a conjugate pair of complex eigenvalues of A, then their eigenvectors also occur in conjugate pairs, i.e. if

$$\vec{v} = \vec{a} + i\,\vec{b}$$

(where $\vec{a}, \vec{b}$ are real vectors) is an eigenvector for λ, then

$$\overline{\vec{v}} = \vec{a} - i\,\vec{b}$$

is an eigenvector for $\bar{\lambda}$. Therefore

$$\vec{u}(t) = e^{\lambda t}\vec{v} \quad \& \quad \overline{\vec{u}(t)} = e^{\bar{\lambda} t}\,\overline{\vec{v}}$$

are solutions to the system. In terms of real-valued functions:

$$\vec{x}_1(t) = \mathrm{Re}\,\vec{u}(t) = e^{\alpha t}\left(\cos(\beta t)\,\vec{a} - \sin(\beta t)\,\vec{b}\right)$$

$$\vec{x}_2(t) = \mathrm{Im}\,\vec{u}(t) = e^{\alpha t}\left(\sin(\beta t)\,\vec{a} + \cos(\beta t)\,\vec{b}\right)$$

are real-valued linearly indp. solutions.

In conclusion : When $\lambda = \alpha + i\beta$ is a complex eigenvalue w/ eigenvector $\vec{v} = \vec{a} + i\,\vec{b}$, the general solution must contain the terms $c_1\vec{x}_1(t)$ & $c_2\vec{x}_2(t)$, where $\vec{x}_1, \vec{x}_2$ are as above.

$$\vec{u}(t) = e^{\lambda t}\vec{v} = e^{\alpha t + i\beta t}(\vec{a} + i\vec{b})$$

$$= e^{\alpha t}\big(\cos(\beta t) + i\sin(\beta t)\big)(\vec{a} + i\vec{b})$$

$$= e^{\alpha t}\Big[\big(\cos(\beta t)\vec{a} - \sin(\beta t)\vec{b}\big) + i\big(\sin(\beta t)\vec{a} + \cos(\beta t)\vec{b}\big)\Big]$$

Example: $\vec{x}\,' = \begin{pmatrix} 2 & 8 \\ -1 & -2 \end{pmatrix}\vec{x}$

Char. Eqn. of A: $\begin{vmatrix} 2-\lambda & 8 \\ -1 & -2-\lambda \end{vmatrix} = -(2+\lambda)(2-\lambda) + 8$

$= \lambda^2 - 4 + 8 = \underline{\underline{\lambda^2 + 4}}$

=> eigenvalues: $\lambda_1 = 2i$; $\lambda_2 = -2i = \overline{\lambda_1}$

Eigenvector for $\lambda_1 = 2i$?

$$(A - 2iI \mid \vec{0}) = \left(\begin{array}{cc|c} 2-2i & 8 & 0 \\ -1 & -2-2i & 0 \end{array}\right) \Rightarrow \boxed{v_1 = -(2+2i)v_2}$$

This will be it:

$$\begin{cases} (2-2i)v_1 + 8v_2 = 0 & \Rightarrow (1-i)v_1 + 4v_2 = 0 \\ -v_1 - (2+2i)v_2 = 0 & \Rightarrow v_1 = -(2+2i)v_2 \end{cases}$$

$$-(1-i)(2+2i)v_2 + 4v_2 = 0$$

$$-(2 + \cancel{2i} - \cancel{2i} + 2)v_2 + 4v_2 = 0$$

$$-4v_2 + 4v_2 = 0 \quad \underline{\text{True}}$$

=> eigenvector: $\begin{pmatrix} -(2+2i)c \\ c \end{pmatrix}$

$\begin{pmatrix} -2c - 2ci \\ c \end{pmatrix} = \begin{pmatrix} -2c \\ c \end{pmatrix} + \begin{pmatrix} -2ci \\ 0 \end{pmatrix}$

=> choose for example $c = -1$:

$$\vec{v} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} + i\begin{pmatrix} 2 \\ 0 \end{pmatrix} \qquad \lambda = 2i$$

$\Rightarrow$ <u>general</u> sol.: $\alpha = 0,\ \beta = 2;\quad \vec{a} = \begin{pmatrix} -2 \\ 1 \end{pmatrix};\ \vec{b} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

$$\vec{x} = C_1 \left[\cos(2t)\begin{pmatrix} 2 \\ -1 \end{pmatrix} - \sin(2t)\begin{pmatrix} 2 \\ 0 \end{pmatrix} \right]$$

$$+ C_2 \left[\sin(2t)\begin{pmatrix} 2 \\ -1 \end{pmatrix} + \cos(2t)\begin{pmatrix} 2 \\ 0 \end{pmatrix} \right]$$

$$= C_1 \begin{bmatrix} +2\cos(2t) - 2\sin(2t) \\ -\cos(2t) \end{bmatrix} + C_2 \begin{bmatrix} 2\sin(2t) + 2\cos(2t) \\ -\sin(2t) \end{bmatrix}$$