Last time: Homogeneous Linear Systems $\vec{x}' = A\vec{x}$. (A is $n\times n$, real matrix)

- Looked for a solution of the form $\vec{x} = \vec{v}e^{\lambda t}$, where $\vec{v}$ is a vector of constants. We got that such functions are solutions to the system, provided that $(A-\lambda I)\vec{v} = 0$, i.e. λ is an eigenvalue of A w/ eigenvector $\vec{v}$.

Case 1: A has n distinct real eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ w/ eigenvectors $\vec{v_1}, \vec{v_2}, \dots, \vec{v_n}$:

$$\boxed{\vec{x} = c_1\vec{v_1}e^{\lambda_1 t} + c_2\vec{v_2}e^{\lambda_2 t} + \dots + c_n\vec{v_n}e^{\lambda_n t}}$$

(general solution).

Case 2: If A has a complex eigenvalue $\lambda = \alpha + i\beta$, with an eigenvector $\vec{v} = \vec{a} + i\vec{b}$, then the complex functions:

$$\vec{u}(t) = e^{\lambda t}\vec{v} \quad \& \quad \overline{\vec{u}(t)} = e^{\bar{\lambda} t}\overline{\vec{v}}$$

are solutions. Extract the real-valued solutions $\operatorname{Re}\vec{u}(t), \operatorname{Im}\vec{u}(t)$:

$$\vec{u}(t) = e^{\lambda t}\vec{v} = e^{\alpha t + i\beta t}(\vec{a} + i\vec{b})$$

$$= e^{\alpha t}\left(\cos(\beta t) + i\sin(\beta t)\right)(\vec{a} + i\vec{b})$$

$$= e^{\alpha t}\Big[\left(\cos(\beta t)\vec{a} - \sin(\beta t)\vec{b}\right) + i\left(\sin(\beta t)\vec{a} + \cos(\beta t)\vec{b}\right)\Big]$$

$\Rightarrow$ In this case, we must add to our general (<u>real</u>) solution:

$$c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t),$$

where:

$$\vec{x}_1(t) = \operatorname{Re} \vec{u}(t) = e^{\alpha t}\left(\cos(\beta t)\vec{a} - \sin(\beta t)\vec{b}\right)$$

$$\vec{x}_2(t) = \operatorname{Im} \vec{u}(t) = e^{\alpha t}\left(\sin(\beta t)\vec{a} + \cos(\beta t)\vec{b}\right)$$

We saw that for $\vec{x}' = \begin{pmatrix} 2 & 8 \\ -1 & -2 \end{pmatrix}\vec{x}$, we have complex eigenvalues:

$\lambda = 2i$, w/ an eigenvector: $\vec{v} = \underbrace{\begin{pmatrix} 2 \\ -1 \end{pmatrix}}_{\vec{a}} + i\underbrace{\begin{pmatrix} 2 \\ 0 \end{pmatrix}}_{\vec{b}}$

$\alpha = 0, \beta = 2$

So, general solution:

$$\vec{x} = c_1\left[\cos(2t)\begin{pmatrix} 2 \\ -1 \end{pmatrix} - \sin(2t)\begin{pmatrix} 2 \\ 0 \end{pmatrix}\right]$$

$$+ c_2\left[\sin(2t)\begin{pmatrix} 2 \\ -1 \end{pmatrix} + \cos(2t)\begin{pmatrix} 2 \\ 0 \end{pmatrix}\right]$$

Case 3: Repeated Eigenvalue

Example:

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Char. Eqn. $\det(A - \lambda I) = 0$ yields $\boxed{(\lambda-2)(\lambda+1)^2 = 0}$

$\lambda_1 = 2$ $\quad \lambda_2 = \lambda_3 = -1$

Double eigenvalue

$\Rightarrow$ -1 is an eigenvalue w/ algebraic multiplicity 2.

Eigenvectors:

for $\lambda_1 = 2$: $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

for $\lambda_2 = \lambda_3 = -1$: $\underbrace{\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}}$

-1 has 2 linearly indp. eigenvectors, so -1 is an eigenvalue w/ geometric multiplicity 2.

[Detailed calculations of all this @ the end] (☆)

Generally: Let λ be an eigenvalue of the matrix A.

- the **algebraic multiplicity** of λ is the multiplicity of λ as a solution to the char. eqn. $\det(A-\lambda I)=0$.

- the **geometric multiplicity** of λ is the dimension of its eigenspace (# of linearly indp. eigenvectors λ can have).

- If $\text{alg.}(\lambda) = \text{geom.}(\lambda)$, we say the eigenvalue λ is **non-defective.**

Otherwise, if $\text{geom.}(\lambda) < \text{alg.}(\lambda)$, we say λ is **defective.**

A matrix which has a defective eigenvalue is called a **defective matrix.**

Example 1: $\vec{x}' = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \vec{x}$

Char. Eqn.: $\begin{vmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (2-\lambda)^2 = 0 \Rightarrow \lambda = 2$ is a double eigenvalue

$\text{alg}(\lambda=2) = \underline{\underline{2}}$

Eigenvectors?

$(A-2I)\vec{v} = \vec{0}$ $\Rightarrow$ this is always true $(\vec{0}=\vec{0})$ no matter what $\vec{v}$ does!

$\downarrow$

$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

So: any $\vec{v} \in \mathbb{R}^2$ is an eigenvector

($\mathbb{R}^2$ is 2-dimensional) $\Rightarrow \text{geom}(\lambda=2) = \underline{\underline{2}}$

$\Rightarrow$ matrix is non-defective

Choose for example $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and we have two linearly indp. vectors, and lin. indp. solutions:

$$\vec{x}_1 = e^{2t}\begin{pmatrix}1\\0\end{pmatrix} \quad \text{and} \quad \vec{x}_2 = e^{2t}\begin{pmatrix}0\\1\end{pmatrix}$$

=> general solution:

$$\boxed{\vec{x} = c_1 e^{2t}\begin{pmatrix}0\\1\end{pmatrix} + c_2 e^{2t}\begin{pmatrix}1\\0\end{pmatrix}}$$

Example 2: $\vec{x}' = \begin{pmatrix}2 & 1\\0 & 2\end{pmatrix}\vec{x}$

Char. Eqn.: $\begin{vmatrix}2-\lambda & 1\\0 & 2-\lambda\end{vmatrix} = 0 \Rightarrow (2-\lambda)^2 = 0 \Rightarrow \lambda = 2$ double eigenvalue

$alg(\lambda=2) = \underline{\underline{2}}$

Eigenvectors? $(A - 2I \mid \vec{0})$

$$\left(\begin{array}{cc|c}0 & 1 & 0\\0 & 0 & 0\end{array}\right) \Rightarrow \text{one restriction: } v_2 = 0$$

$\Rightarrow$ eigenvector: $\vec{v} = \begin{pmatrix}1\\0\end{pmatrix}$

$geom(\lambda=2) = \underline{\underline{1}}$

$\Downarrow$

defective eigenvalue!

This only gives us one solution:

$$\vec{x}_1 = e^{2t}\begin{pmatrix}1\\0\end{pmatrix}$$

So we only have $c_1 e^{2t}\begin{pmatrix}1\\0\end{pmatrix}$ to put in our general solution.

How do we find a second solution?

Suppose A is a 2×2 matrix and λ is an eigenvalue w/ algebraic multiplicity 2, and geometric multiplicity 1.

To solve $\vec{x}' = A\vec{x}$:

1. Find an eigenvector $\vec{v}$ corresponding to λ

$$\Rightarrow \boxed{\vec{x}_1 = e^{\lambda t}\vec{v}}$$

2. Second solution: a solution of the form:

$$\vec{x}_2 = te^{\lambda t}\vec{v} + e^{\lambda t}\vec{w}$$

can be found, where $\vec{v}$ is the same eigenvector as in 1, and $\vec{w}$ is a vector that satisfies:

$$(A-\lambda I)\vec{w} = \vec{v} \qquad (*)$$

$\Rightarrow$ General solution:

$$\boxed{\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2}$$

Pf: Let $\vec{x}_2 = te^{\lambda t}\vec{v} + e^{\lambda t}\vec{w}$ $\qquad A\vec{v} = \lambda\vec{v}$

$$\Rightarrow \vec{x}_2' = e^{\lambda t}\vec{v} + t\lambda e^{\lambda t}\vec{v} + \lambda e^{\lambda t}\vec{w}$$

$$\Rightarrow A\vec{x}_2 = te^{\lambda t}\underbrace{A\vec{v}}_{=\lambda\vec{v}} + e^{\lambda t}\underbrace{A\vec{w}}_{=\lambda\vec{w}+\vec{v}} = te^{\lambda t}\lambda\vec{v} + \lambda e^{\lambda t}\vec{w} + e^{\lambda t}\vec{v}$$

$\lambda\vec{v}$ b/c $(\lambda, \vec{v})$ is an eigenpair

$\lambda\vec{w}+\vec{v}$ by $(*)$

$$\Rightarrow \boxed{\vec{x}_2' = A\vec{x}_2}$$

∎

Back to example 2: $\vec{x}' = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \vec{x}$

We found the double eigenvalue $\lambda = 2$, and one eigenvector $\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\Rightarrow$ $\vec{x}_1 = e^{2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Find $\vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ s.t. $(A - 2I)\vec{w} = \vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{cc|c} 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow w_2 = 1$$

no restrictions on w_1

$\Rightarrow$ $\vec{w}$ can be any vector of the form $\begin{pmatrix} c \\ 1 \end{pmatrix}$ with $c \in \mathbb{R}$.

Take, for example, $c = 0$:

$$\vec{w} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow \vec{x}_2 = te^{2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

General Solution: $\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2 = \begin{pmatrix} c_1 e^{2t} \\ 0 \end{pmatrix} + \begin{pmatrix} c_2 t e^{2t} \\ c_2 e^{2t} \end{pmatrix}$

$$= e^{2t}\begin{pmatrix} c_1 + c_2 t \\ c_2 \end{pmatrix}$$

Detailed calculations for ☆:

Find eigenvalues & eigenvectors: $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

Char. Eqn.: $\begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0$

$$-\lambda \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 1 & -\lambda \end{vmatrix} + \begin{vmatrix} 1 & -\lambda \\ 1 & 1 \end{vmatrix} = 0$$

$$-\lambda(\lambda^2 - 1) - (-\lambda - 1) + (1 + \lambda) = 0$$

$$-\lambda^3 + \lambda + \lambda + 1 + 1 + \lambda = 0$$

$$-\lambda^3 + 3\lambda + 2 = 0$$

$$-\lambda^3 - \lambda^2 + \lambda^2 + 3\lambda + 2 = 0$$

$$-\lambda^2(\lambda+1) + (\lambda+2)(\lambda+1) = 0$$

$$(-\lambda^2 + \lambda + 2)(\lambda+1) = 0$$

$$(\lambda^2 - \lambda - 2)(\lambda+1) = 0$$

$$(\lambda-2)(\lambda+1)(\lambda+1) = 0$$

double eigenvalue

$$(\lambda-2)(\lambda+1)^2 = 0 \Rightarrow \lambda = 2 \quad \lambda = -1$$

$\lambda = 2$: $\left(\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right) \xrightarrow[R_3 \times 2]{R_2 \times 2} \left(\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 2 & -4 & 2 & 0 \\ 2 & 2 & -4 & 0 \end{array}\right)$

$\xrightarrow[R_3 - R_1]{R_2 - R_1} \begin{pmatrix} -2 & 1 & 1 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{pmatrix} \xrightarrow[R_3 : (3)]{R_2 : (-3)} \left(\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right)$

$$\xrightarrow[R_2-R_2]{R_1-R_2} \left(\begin{array}{ccc|c} -2 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1:(-2)} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{cases} v_1 - v_3 = 0 \\ v_2 - v_3 = 0 \end{cases} \qquad \begin{cases} v_1 = v_3 \\ v_2 = v_3 \end{cases}$$

$$\Rightarrow \begin{pmatrix} c \\ c \\ c \end{pmatrix} \Rightarrow \underline{\text{eigenvector}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\lambda = -1: \left(\begin{array}{ccc|c} +1 & 1 & 1 & 0 \\ 1 & +1 & 1 & 0 \\ 1 & 1 & +1 & 0 \end{array}\right) \Rightarrow \underline{\text{one equation}}: \boxed{v_1 + v_2 + v_3 = 0}$$

=> 2 of the values can be chosen arbitrarily, and the third is determined by them

$$\vec{v} = \begin{pmatrix} c_1 \\ c_2 \\ -c_1 - c_2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$\Rightarrow \underline{\text{eigenvectors}}: \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \quad \underline{\underline{\text{linearly indp.}}}$$