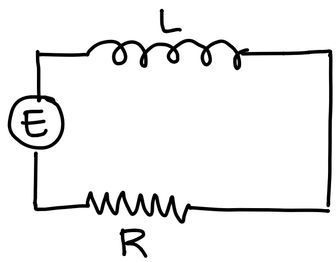
ODEs w/ Discontinuous Forcing Functions + Series in ODEs

Example: L-R Series Circuit



Series circuit containing a resistor R and an inductor L:

$i(t)$ = current at time t

$E(t)$ = voltage impressed on the circuit

$L\frac{di}{dt}$ = voltage drop across the inductor

iR = voltage drop across the resistor

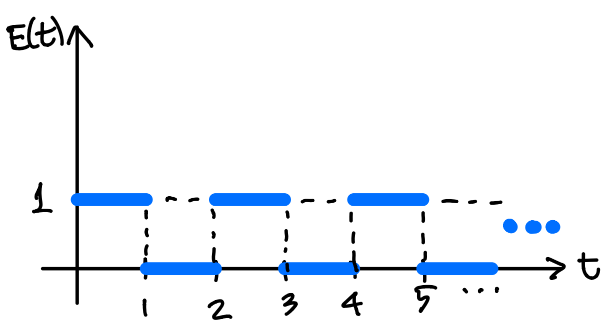
Kirchhoff's Second Law:

$$\boxed{L\frac{di}{dt} + Ri = E(t)}$$

→ a differential equation for the current $i(t)$

Solving this depends on how complicated the voltage $E(t)$ is. Often times, $E(t)$ is discontinuous.

① Find the current $i(t)$ in a single loop L-R series circuit, given that the voltage $E(t)$ is the square wave below, and the initial condition $i(0)=0$.



$E(t)$ = periodic w/ period $T=2$

$$\mathcal{L}\{E(t)\} = \frac{1}{1-e^{-2s}}\int_0^2 e^{-st}E(t)\,dt$$

$$= \frac{1}{1-e^{-2s}}\int_0^1 e^{-st}\,dt$$

$$= \frac{1}{1-e^{-2s}}\left(\frac{-1}{s}e^{-st}\right)\Big|_{t=0}^{1}$$

$$= \frac{1}{1-e^{-2s}}\left(\frac{-1}{s}e^{-s} + \frac{1}{s}\right) = \frac{1-e^{-s}}{s(1-e^{-2s})}$$

$$= \frac{1-e^{-s}}{s(1-e^{-s})(1+e^{-s})} = \frac{1}{s(1+e^{-s})}$$

$$\boxed{\mathcal{L}\{E(t)\} = \frac{1}{s(1+e^{-s})}}$$

Laplace transform of ODE: (remember $i(t)$ is the unknown function here, and L & R are constants)

$$Li' + Ri = E(t)$$

$$L\,\mathcal{L}\{i'\} + R\,\mathcal{L}\{i\} = \mathcal{L}\{E(t)\}$$

$$L\left(sI(s) - \underbrace{i(0)}_{0} + RI(s)\right) = \frac{1}{s(1+e^{-s})}$$

$$(Ls+R)I(s) = \frac{1}{s(1+e^{-s})} \Rightarrow \boxed{I(s) = \frac{1}{(Ls+R)s(1+e^{-s})}}$$

$$\Rightarrow \boxed{i(t) = \mathcal{L}^{-1}\left\{\frac{1}{s(Ls+R)(1+e^{-s})}\right\}}$$

Now, how to find this inverse Laplace?

- The part with $s(Ls+R)$ we could do partial fractions for:

$$\frac{1}{s(Ls+R)} = \frac{A}{s} + \frac{B}{Ls+R} = \frac{1}{Rs} - \frac{L}{R(Ls+R)}$$

$$1 = A(Ls+R) + Bs \qquad s=0: 1 = AR \Rightarrow A = 1/R$$

$$s = \frac{-R}{L}: 1 = \frac{-R}{L}B \Rightarrow B = -L/R$$

- So now we have:

$$i(t) = \frac{1}{R}\mathcal{L}^{-1}\left\{\frac{1}{s}\frac{1}{1+e^{-s}}\right\} - \frac{1}{R}\mathcal{L}^{-1}\left\{\frac{1}{s+R/L}\,\frac{1}{1+e^{-s}}\right\}$$

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = \mathcal{L}^{-1}\{F(s)\}\Big|_{t\to t-a} u_a(t)$$

Problem: We know how to handle things like $\mathcal{L}^{-1}\{e^{-as}F(s)\}$, but how to deal with the term $\frac{1}{1+e^{-s}}$?! Series Expansions!

Recall from Calculus II: (the geometric series):

$$\boxed{\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots = \sum_{k=0}^{\infty}(-1)^k x^k \quad \text{for all } x \in \mathbb{R} \text{ such that } |x|<1.}$$

So let $x = e^{-s}$ above, with $s>0$:

$$\frac{1}{1+e^{-s}} = \sum_{k=0}^{\infty}(-1)^k (e^{-s})^k = \sum_{k=0}^{\infty}(-1)^k e^{-sk}$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s}\frac{1}{1+e^{-s}}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\sum_{k=0}^{\infty}(-1)^k e^{-sk}\right\}$$

$$= \sum_{k=0}^{\infty}(-1)^k \mathcal{L}^{-1}\left\{\frac{1}{s}e^{-sk}\right\}$$

$$= \sum_{k=0}^{\infty}(-1)^k \underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}\Big|_{t\to t-k}}_{1|_{t\to t-k}=1} u_k(t)$$

$$= \sum_{k=0}^{\infty}(-1)^k u_k(t).$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s+R/L}\,\frac{1}{1+e^{-s}}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s+R/L}\sum_{k=0}^{\infty}(-1)^k e^{-sk}\right\}$$

$$= \sum_{k=0}^{\infty}(-1)^k \mathcal{L}^{-1}\left\{\frac{1}{s+R/L}\, e^{-sk}\right\}$$

$$= \sum_{k=0}^{\infty}(-1)^k \underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s+R/L}\right\}\Big|_{t\to t-k}}_{e^{-R/L\, t}\big|_{t\to t-k} = e^{-\frac{R}{L}(t-k)}} u_k(t)$$

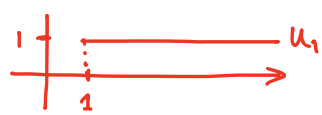
$$= \sum_{k=0}^{\infty}(-1)^k e^{-\frac{R}{L}(t-k)} u_k(t)$$

$$\Rightarrow i(t) = \frac{1}{R}\sum_{k=0}^{\infty}(-1)^k u_k(t) - \frac{1}{R}\sum_{k=0}^{\infty}(-1)^k e^{-\frac{R}{L}(t-k)} u_k(t)$$

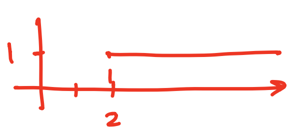
$$\boxed{i(t) = \frac{1}{R}\sum_{k=0}^{\infty}(-1)^k\left(1-e^{-\frac{R}{L}(t-k)}\right)u_k(t)}$$

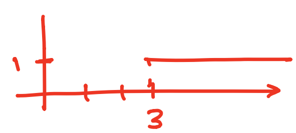
Take $R = L = 1$:

$$i(t) = \left(1-e^{-t}\right)u_0(t) - \left(1-e^{-(t-1)}\right)u_1(t)$$

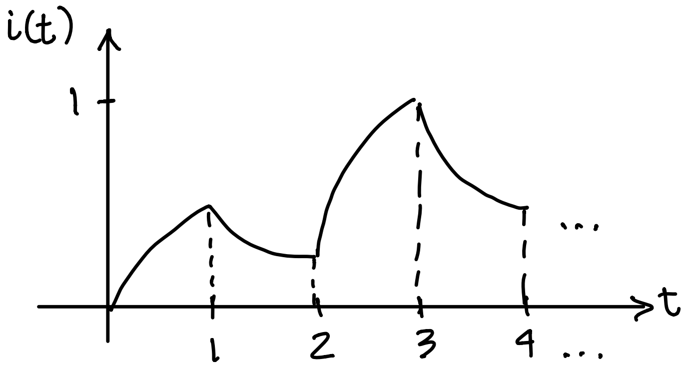


$$+\left(1-e^{-(t-2)}\right)u_2(t) - \left(1-e^{-(t-3)}\right)u_3(t) + \dots$$





$$
i(t) = \begin{cases} 0 \le t < 1: & 1 - e^{-t} \\ 1 \le t < 2: & (1 - e^{-t}) - (1 - e^{-(t-1)}) = -e^{-t} + e^{-(t-1)} \\ 2 \le t < 3: & (1 - e^{-t}) - (1 - e^{-(t-1)}) + (1 - e^{-(t-2)}) \\ & = -e^{-t} + e^{-(t-1)} + 1 - e^{-(t+2)} \\ \quad \vdots & \end{cases}
$$



Review of Power Series

- **Infinite Series:** $\sum_{n=1}^{\infty} a_n$

  - **Sequence of partial sums:** $S_N = \sum_{n=1}^{N} a_n$

$$S_1 = a_1$$
$$S_2 = a_1 + a_2$$
$$S_3 = a_1 + a_2 + a_3$$
$$\vdots$$

  - The sequence $\sum_{n=1}^{\infty} a_n$ is said to converge to some number S if and only if

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty}\left(\sum_{n=1}^{N} a_n\right) = S.$$

- **The geometric series:**

$$a + ar + ar^2 + ar^3 + \ldots = \sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, \text{ iff } |r|<1$$

(divergent if $|r| \geq 1$).

- **Transition to variables:**

Example: For what values of x does the series $\sum_{n=1}^{\infty} (-2)^n x^n$ converge?

$$\sum_{n=1}^{\infty} (-2x)^n = \sum_{n=1}^{\infty} (-2x)\cdot(-2x)^{n-1} \quad \text{is geometric w/}$$
$$a = r = -2x$$

$\Rightarrow$ Series converges $\Leftrightarrow$ $|-2x| = |2x| < 1$ $\Leftrightarrow$ $|x| < \frac{1}{2}$

$$\boxed{x \in (-1/2, 1/2)}$$

• Power Series: A power series centered at a (about a), where a is a constant, is a series of the form:

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots$$

There are 3 possibilities: (radius (R) and interval (I) of convergence)

1. Series only converges at $x=a$: $R=0$; $I=\{a\}$.
2. Series converges for all real x: $R=\infty$; $I=\mathbb{R}=(-\infty,\infty)$
3. There is some $0<R<\infty$ such that the series converges absolutely for all $|x-a|<R$ $(a-R<x<a+R)$ and diverges for $|x-a|>R$ (anything can happen at the endpoints $a-R$ & $a+R$!)

   I could be $(a-R, a+R)$, $(a-R, a+R]$, $[a-R, a+R)$, or $[a-R, a+R]$.

• A power series defines a function $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ (domain = interval of convergence).

• Point: go backwards, i.e. start w/ a function $f(x)$ and express it as a power series about some $x=a$ (power series expansion of f).

Def: A function $f(x)$ is said to be **analytic at a point $x=a$** if it can be represented by a power series centered at a, with radius of convergence $R>0$ (could be ∞).

• In Calc 2: infinitely differentiable functions such as e^x, $\sin(x)$, $\cos(x)$, $\ln(1+x)$ etc. can be represented by Taylor series:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

or by a Maclaurin series (Taylor expansion at $a=0$):

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

• Some familiar (hopefully) examples:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \qquad (x \in \mathbb{R})$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \qquad (x \in \mathbb{R})$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \qquad (x \in \mathbb{R})$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}x^n = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \qquad (x \in (-1,1])$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots \qquad (x \in (-1,1))$$

Recall that you can use these to obtain new power series:

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$$

$$\ln(x) = \ln(1+(x-1)) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n, \qquad x-1 \in (-1,1]$$

$$\underline{\underline{x \in (0,2]}}$$