Power Series Solutions to ODEs

A preview example: Find a power series solution to

$$y' + y = 0$$

Assume the solution y is a power series centered at 0

$$y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots$$

$$\Rightarrow y' = c_1 + 2c_2 x + 3c_3 x^2 + \ldots = \sum_{n=1}^{\infty} c_n n x^{n-1}$$

Substitute in ODE:

$$y' + y = \underbrace{\sum_{n=1}^{\infty} c_n \cdot n x^{n-1}} + \sum_{n=0}^{\infty} c_n x^n = 0$$

Re-index this sum to start it at 0: $n \mapsto n+1$

$$\sum_{n=0}^{\infty} c_{n+1} (n+1) x^n$$

$$\sum_{n=0}^{\infty} c_{n+1} (n+1) x^n + \sum_{n=0}^{\infty} c_n x^n = 0$$

Combine:

$$\boxed{\sum_{n=0}^{\infty} \left[c_{n+1} (n+1) + c_n \right] x^n = 0}$$

We want this to be true for all x in some interval, so the only way that can happen is to have

$$c_{n+1}(n+1) + c_n = 0$$

$$\boxed{c_{n+1} = \frac{-c_n}{n+1}, \quad \forall n = 0, 1, 2, \ldots}$$

recurrence relationship

recurrence relationship

$$\boxed{C_{n+1} = \frac{-C_n}{n+1}, \quad \forall n=0,1,2,\dots}$$

$$C_0 = (-1)^0 \cdot \frac{C_0}{0!}$$

$n=0$: $C_1 = \frac{-C_0}{0+1} = -C_0 = (-1)^1 \cdot \frac{C_0}{1!}$

$n=1$: $C_2 = \frac{-C_1}{1+1} = \frac{-C_1}{2} = \frac{-(-C_0)}{2} = \frac{C_0}{2} = (-1)^2 \cdot \frac{C_0}{2!}$

$n=2$: $C_3 = \frac{-C_2}{2+1} = -\frac{C_2}{3} = -\frac{C_0/2}{3} = -\frac{C_0}{2\cdot 3} = (-1)^3 \frac{C_0}{3!}$

$n=3$: $C_4 = \frac{-C_3}{3+1} = \frac{-C_3}{4} = \frac{+C_0/2\cdot 3}{4} = \frac{C_0}{1\cdot 2\cdot 3\cdot 4} = (-1)^4 \frac{C_0}{4!}$

Recall these are the coefficients of our solution y!

$$y(x) = \sum_{n=0}^{\infty} C_n x^n = \sum_{n=0}^{\infty} (-1)^n \frac{C_0}{n!} x^n$$

$$\boxed{y = C_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n}$$

$$C_0 \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = C_0 e^{-x}$$

Series Solutions to ODEs about ordinary points.

Homogeneous linear 2nd order ODE:

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

Standard form: (divide by $a_2(x) \neq 0$)

$$\boxed{y'' + P(x)y' + Q(x)y = 0} \qquad (*)$$

Def: A point $x=a$ is an
- ordinary point of the ODE if both coefficients $P(x), Q(x)$ are analytic at $x=a$.
- singular point of the ODE otherwise.

Ex): $y'' + 3y' + 2y = 0$ A homogeneous linear ODE w/ constant coefficients has no singular points.

Ex): $y'' + e^x y' + (\sin x) y = 0$ also has no singular points (b/c both e^x and $\sin x$ have Taylor expansions at all $a \in \mathbb{R}$).

Ex): $y'' + xy' + (\ln x) y = 0$ has a singular point at $x = 0$ because $Q(x) = \ln x$ is not analytic at 0 (it is not even defined at 0).

Our main concern: polynomial coefficients: if all $a_2(x), a_1(x), a_0(x)$ are polynomials, the standard form coefficients are all rational functions $\left(P(x) = \dfrac{a_1(x)}{a_2(x)} ; Q(x) = \dfrac{a_0(x)}{a_2(x)} \right)$

analytic at all points except where the denominator is 0!

Ex): $(x^2 - 1) y'' + 2xy' + 6y = 0$

Standard Form: $y'' + \dfrac{2x}{x^2 - 1} y' + \dfrac{6}{x^2 - 1} y = 0$

Singular points: $x = \pm 1$ (all other are ordinary)

Ex): $(x^2 + 1) y'' + xy' - y = 0$

Standard Form: $y'' + \dfrac{x}{x^2 + 1} y' - \dfrac{1}{x^2 + 1} y = 0$

All $x \in \mathbb{R}$ are ordinary points!

The only singular points are complex $(x = \pm i)$

Theorem: If $x=a$ is an ordinary point for the (standard) equation

$$y'' + P(x)y' + Q(x)y = 0$$

we can always find two linearly independent solutions in the form of a power series centered at a, i.e.:

$$y = \sum_{n=0}^{\infty} c_n (x-a)^n$$

A power series solution will converge at least on some interval defined by $|x-a| < R$, where R is the distance from a to the closest singular point.

Finding a power series solution (centered at 0) to:

$$a_2(x)y''(x)+a_1(x)y'(x)+a_0(x)y(x)=0$$

1. Suppose $\boxed{y=\sum_{n=0}^{\infty}c_n x^n} \Rightarrow \boxed{y'=\sum_{n=0}^{\infty}(n+1)c_{n+1}x^n}$

and $\boxed{y''=\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n}$

2. Replace these in the original ODE and re-index as needed.

$$y=c_0+c_1x+c_2x^2+c_3x^3+\dots$$

$$y'=c_1+2c_2x+3c_3x^2+\dots=\sum_{n=1}^{\infty}nc_nx^{n-1}\underset{\text{re-index}}{=}\sum_{n=0}^{\infty}(n+1)c_{n+1}x^n$$

$$y''=2c_2+3\cdot 2c_3x+4\cdot 3c_4x^2+\dots=\sum_{n=2}^{\infty}n(n-1)c_nx^{n-2}$$

re-index $\rightarrow$ $=\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n$

or $\rightarrow$ $=\sum_{n=1}^{\infty}(n+1)nc_{n+1}x^{n-1}$

3. The identity function $(x)=0$ results in making all coefficients on the left-hand side equal to 0:

$$\sum\underbrace{[\text{stuff in } c_n, c_{n-1}, c_{n+1}\dots]}_{=0}x^n=0$$

4. This gives a recurrence relationship b/w coefficients.
In the end, everything will be in terms of the first two coefficients c_0 & c_1, and the general solution $y=c_0y_1+c_1y_2$.

Example: Airy's Equation (appears in applications involving light diffraction, radio wave diffraction, aerodynamics & more):

$$\boxed{y'' + xy = 0}$$

No singular points => Can find power series solution at any point but let's stick to Maclaurin series (centered at 0):

$$y(x) = \sum_{n=0}^{\infty} c_n x^n \qquad \text{(Goal: find the coefficients } c_n\text{)}$$

$$\Rightarrow \quad y'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

$$\Rightarrow \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

Substitute in ODE:

$$0 = y'' + xy = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + x \cdot \sum_{n=0}^{\infty} c_n x^n$$

$$= \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+1}$$

$$2c_2 + 3\cdot 2 c_3 x + 4\cdot 3 c_4 x^2 + \ldots \qquad c_0 x + c_1 x^2 + c_2 x^3 + \ldots$$

This indicates we should re-index starting at $n=1$, and separate out the $2c_2$

$$= 2c_2 + \sum_{n=1}^{\infty} (n+2)(n+1) c_{n+2} x^n + \sum_{n=1}^{\infty} c_{n-1} x^n$$

$$= 2c_2 + \sum_{n=1}^{\infty} \left[(n+2)(n+1) c_{n+2} + c_{n-1} \right] x^n = 0$$

$$2c_2 + \sum_{n=1}^{\infty}\left[(n+2)(n+1)c_{n+2} + c_{n-1}\right]x^n = 0$$

$$\Rightarrow \begin{cases} 2c_2 = 0 \\ (n+2)(n+1)c_{n+2} + c_{n-1} = 0 \end{cases} \Rightarrow \boxed{c_{n+2} = \frac{-c_{n-1}}{(n+2)(n+1)}}$$

Recurrence Relationship $n = 1, 2, 3, \ldots$

c_0

c_1

$c_2 = 0$

$n=1$: $c_3 = \dfrac{-c_0}{3\cdot 2}$

$n=2$: $c_4 = \dfrac{-c_1}{4\cdot 3}$

$n=3$: $c_5 = \dfrac{-c_2}{5\cdot 4} = \boxed{0}$ b/c $\boxed{c_2 = 0}$

$n=4$: $c_6 = \dfrac{-c_3}{6\cdot 5} = -\dfrac{-c_0/3\cdot 2}{6\cdot 5} = \dfrac{c_0}{(3\cdot 2)\cdot(6\cdot 5)}$

$n=5$: $c_7 = \dfrac{-c_4}{7\cdot 6} = \dfrac{-c_1/4\cdot 3}{7\cdot 6} = \dfrac{-c_1}{(4\cdot 3)(7\cdot 6)}$

$n=6$: $c_8 = \dfrac{-c_5}{8\cdot 7} = \boxed{0}$ b/c $c_5 = 0$

$n=7$: $c_9 = \dfrac{-c_6}{9\cdot 8} = -\dfrac{c_0/(3\cdot 2)(6\cdot 5)}{9\cdot 8} = \dfrac{-c_0}{(3\cdot 2)(6\cdot 5)(9\cdot 8)}$

$n=8$: $c_{10} = \dfrac{-c_7}{10\cdot 9} = -\dfrac{-c_1/(4\cdot 3)(7\cdot 6)}{10\cdot 9} = \dfrac{c_1}{(4\cdot 3)(7\cdot 6)(10\cdot 9)}$

$$\Rightarrow \quad y = c_0 + c_1 x + 0 - \frac{c_0}{2\cdot 3}x^3 - \frac{c_1}{3.4}x^4 + 0 + \frac{c_0}{(2\cdot 3)(5\cdot 6)}x^6 + \frac{c_1}{(3.4)(6.7)}x^7 + 0 - \frac{c_0}{(2.3)(5.6)(8.9)}x^9 - \frac{c_1}{(3.4)(6.7)(9.10)}x^{10} + 0 - \cdots$$

Group the c_0 terms and the c_1 terms:

$$y(x) = c_0 y_1(x) + c_1 y_2(x)$$

where:

$$\begin{array}{ccc} n=1 & n=2 & n=3 \end{array}$$

$$y_1(x) = 1 - \frac{1}{2\cdot 3}x^3 + \frac{1}{(2\cdot 3)(5\cdot 6)}x^6 - \frac{1}{(2\cdot 3)(5\cdot 6)(8\cdot 9)}x^9 + \ldots$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2\cdot 3)(5\cdot 6)\cdot\ldots\cdot[(3n-1)\cdot 3n]} x^{3n}$$

$$\begin{array}{ccc} n=1 & n=2 & n=3 \end{array}$$

$$y_2(x) = x - \frac{1}{3\cdot 4}x^4 + \frac{1}{(3.4)(6.7)}x^7 - \frac{1}{(3.4)(6.7)(9.10)}x^{10} + \ldots$$

$$= x + \sum_{n=1}^{\infty} \frac{(-1)^n}{(3.4)(6.7)\ldots[(3n)(3n+1)]} x^{3n+1}$$

Remark: c_0 and c_1 can be chosen arbitrarily, so the linear combination $y = c_0 y_1 + c_1 y_2$ is the general solution.

Remark: You won't always be able to find a "summation formula" for all coefficients.