More on Separable Equations

Example: $\dfrac{dy}{dx} = \dfrac{x^2}{1-y^2}$

We "cross-multiply" and treat the differentials algebraically

$$\int (1-y^2)\,dy = \int x^2\,dx$$

$$y - \frac{y^3}{3} = \frac{x^3}{3} + C$$

But what is actually going on and why does this work?

Rewrite the original as:

$$\frac{dy}{dx} = \frac{x^2}{1-y^2}$$

$$(1-y^2)\frac{dy}{dx} = x^2$$

$$\boxed{\underbrace{-x^2}_{\substack{\text{derivative of} \\ -\frac{x^3}{3} \\ \text{wrt } x}} + \underbrace{(1-y^2)\frac{dy}{dx}}_{\substack{\text{derivative of} \\ \left(y-\frac{y^3}{3}\right) \\ \text{wrt } x}} = 0.}$$

Now: Remember the chain rule: if $y(x)$ (y depends on x) how do we differentiate a function $f(y)$ with respect to x?

$$\frac{d}{dx} f(y) = \frac{df}{dy}\frac{dy}{dx} = f'(y)\frac{dy}{dx}$$

For example: $f(y) = y - \dfrac{y^3}{3}$

$$\Rightarrow \frac{d}{dx}\left(y - \frac{y^3}{3}\right) = (1-y^2)\frac{dy}{dx}$$

$$\frac{d}{dx}\left(-\frac{x^3}{3} + y - \frac{y^3}{3}\right) = 0$$

$$\boxed{-\frac{x^3}{3} + y - \frac{y^3}{3} = C}$$

④ $xy^4\,dx + (y^2+2)e^{-3x}\,dy = 0$

$$xy^4\,dx = -(y^2+2)e^{-3x}\,dy \quad /\cdot e^{3x}\,\frac{1}{y^4}$$

(! Division by y,)

$$\int xe^{3x}\,dx = -\int \frac{y^2+2}{y^4}\,dy$$

$$\frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} = \frac{1}{y} + \frac{2}{3y^3} + C$$

$$\boxed{\left(\frac{1}{3}x - \frac{1}{9}\right)e^{3x} - \frac{1}{y} - \frac{2}{3y^3} = C} \quad (*)$$

$\int xe^{3x}\,dx$ $\left(\begin{array}{l} u = x \quad dv = e^{3x} \\ du = dx; \quad v = \frac{1}{3}e^{3x}\end{array}\right)$

$$= \frac{1}{3}xe^{3x} - \int \frac{1}{3}e^{3x}\,dx$$

$$= \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + C$$

$$\int \frac{y^2+2}{y^4}\,dy = \int \frac{1}{y^2} + \frac{2}{y^4}\,dy$$

$$= -\frac{1}{y} - \frac{2}{3}y^{-3} + C$$

Did we lose a solution??

- Is $y=0$ a solution?

$$xy^4 + (y^2+2)e^{-3x}\frac{dy}{dx} = 0 \qquad \text{Yes: } y \equiv 0 \Rightarrow \frac{dy}{dx} = 0$$

- Is $y=0$ represented in $(*)$? <u>No</u>.

<u>Full Solution:</u>

$$\boxed{\begin{array}{c}\left(\frac{1}{3}x - \frac{1}{9}\right)e^{3x} - \frac{1}{y} - \frac{2}{3y^3} = C \\ \underline{\text{and}}\ \ y = 0\end{array}}$$

Generic ODE of order n: $f(t, y, y', \dots, y^{(n)}) = 0$ (1)

A solution to (1) <u>on the interval</u> $t \in (a,b)$ is a function $y = \phi(t)$ such that $\phi', \phi'', \dots, \phi^{(n)}$ exist and satisfy (1):

$$f(t, \phi(t), \phi'(t), \dots, \phi^{(n)}(t)) = 0$$

for <u>every</u> $t \in (a,b)$.

5 $\dfrac{dy}{dx} = \dfrac{3x^2+4x+2}{2(y-1)}$; $y(0) = -1$

Solve and determine the interval on which the solution exist. (aka "interval of validity")

$$\int 2(y-1)\,dy = \int (3x^2+4x+2)\,dx$$

$$y^2 - 2y = x^3 + 2x^2 + 2x + C$$

IVP: $x=0,\ y=-1 \Rightarrow 1+2 = 0 + C \Rightarrow C = 3$

$$\boxed{y^2 - 2y = x^3 + 2x^2 + 2x + 3}$$

Can we solve explicitly?

Add 1 to both sides to make a square on the left-hand side:

$$y^2 - 2y + 1 = x^3 + 2x^2 + 2x + 4$$

$$(y-1)^2 = x^3 + 2x^2 + 2x + 4$$

$$y - 1 = \pm\sqrt{x^3 + 2x^2 + 2x + 4}$$

$$\boxed{y = 1 \pm \sqrt{x^3 + 2x^2 + 2x + 4}}$$

$\rightarrow$ Two solutions ?? But only one passes thru $(0,-1)$.

$$-1 = 1 \pm \sqrt{4} = 1 \pm 2 \Rightarrow \text{Choose } -$$

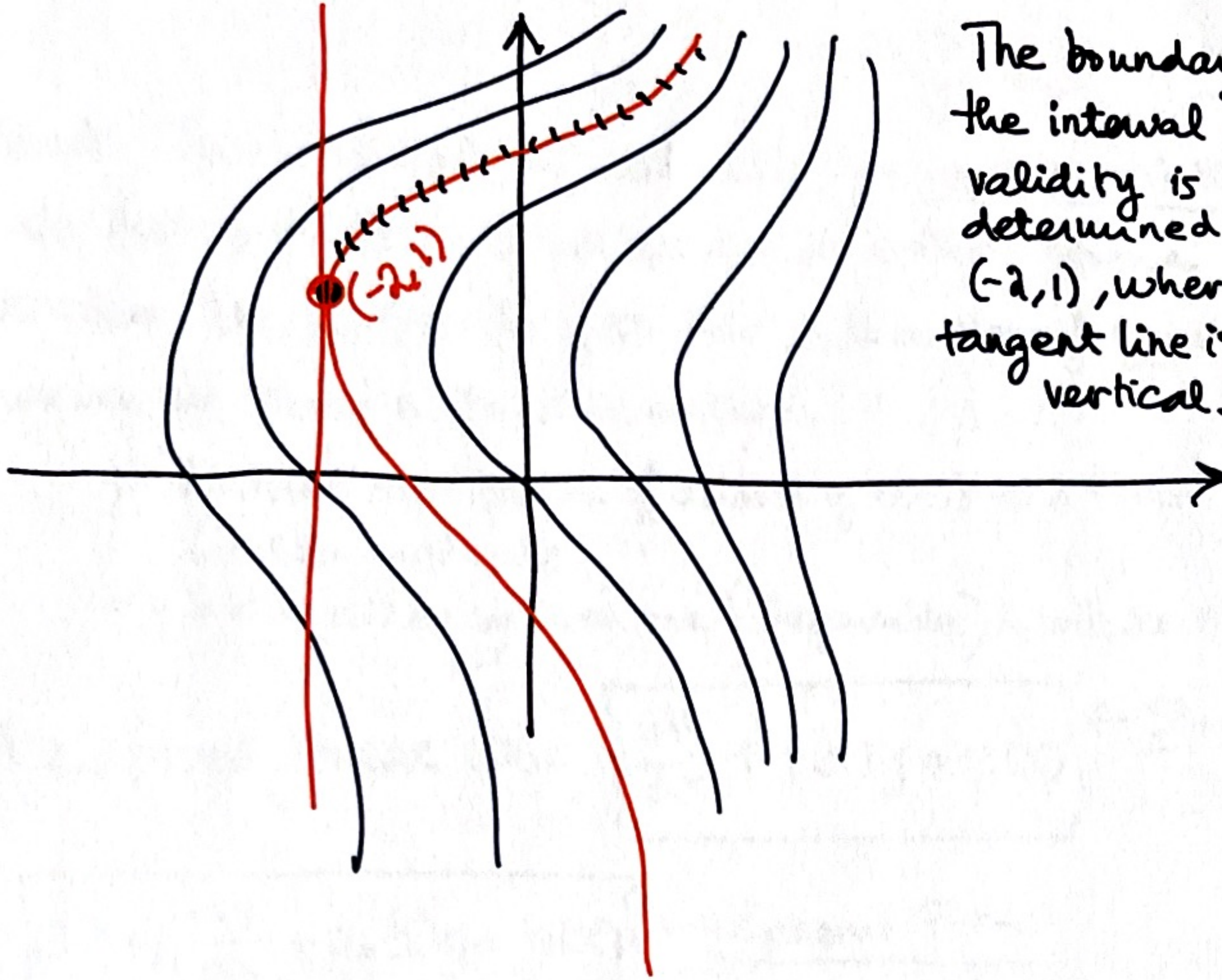
$$\boxed{y = 1 - \sqrt{x^3 + 2x^2 + 2x + 4}}$$

Interval of validity? Find x s.t. $x^3 + 2x^2 + 2x + 4 \geq 0$.

$$x^3 + 2x^2 + 2x + 4 = x^2(x+2) + 2(x+2) = (x+2)(x^2+2)$$

positive for all $x \geq -2$

$\Rightarrow$ Interval of validity: $x \in (-2, \infty)$



The boundary of the interval of validity is determined by (-2,1), where the tangent line is vertical.

2.1. | Linear Eqns.: Integrating Factors

Remark: There is no general method to find <u>analytic solutions</u> for all first order DE's. What we can do is this: given a first order ODE, determine if it belongs to a class of equations for which we know a solution method.

=> We need a collection of classes of ODE's and their solution methods.

=> We've already seen one (Separable) & will see more.

FIRST ORDER LINEAR ODE: $\boxed{\dfrac{dy}{dx} + P(x)y = g(x)}$ ← Standard Form

$\boxed{a_1(x)\dfrac{dy}{dx} + a_0(x)y = h(x)}$ ← General Form

→ Can always be put in Standard Form by dividing by $a_1(x)$ [provided $a_1(x) \neq 0$]

- $g(x) = 0$ => homogeneous : $\dfrac{dy}{dx} + P(x)y = 0$
- $g(x) \neq 0$ => non-homogeneous