# 2.1 Integrating Factors

**Example:** $\frac{dy}{dx} + y = x$ Remark: NOT separable.

Want: multiply the equation by a function $\mu(x)$ called an integrating factor:

$$\underbrace{\mu(x)\frac{dy}{dx}} + \mu(x)y(x) = \mu(x)\cdot x$$

Reminds of Product Rule for derivatives:

$$\frac{d}{dx}[\mu(x)y(x)] = \mu(x)\frac{dy}{dx} + \frac{d\mu}{dx}y(x)$$

To "match up" the two: Is there a function s.t. $\mu = \frac{d\mu}{dx}$?
A function equal to its derivative? Yes: $\mu(x) = e^x$.
So let's try:

$$\frac{dy}{dx} + y = x \quad \Big|\cdot e^x$$

$$\underbrace{e^x\frac{dy}{dx} + e^x y} = xe^x$$

$$= \frac{d}{dx}(e^x \cdot y(x))$$

$$\Rightarrow \frac{d}{dx}(e^x \cdot y) = xe^x$$

$$\Rightarrow e^x \cdot y = \int xe^x dx$$

$$= xe^x - e^x + C$$

$u = x;\ dv = e^x$
$du = dx;\ v = e^x$
$\int xe^x dx = xe^x - \int e^x dx$
$= xe^x - e^x + C$

$$e^x \cdot y = xe^x - e^x + C$$

$$\boxed{y = x - 1 + Ce^{-x}}$$

Check: $\frac{dy}{dx} = 1 - ce^{-x}$

$\Rightarrow \frac{dy}{dx} + y = \cancel{1} - \cancel{ce^{-x}} + (x - \cancel{1} + \cancel{ce^{-x}}) = x$ Yes

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Alternate Solution to $\frac{dy}{dx} + y = x$ (by Substitution):

Rewrite as: $\frac{dy}{dx} = x - y$

Make new variable $u = x - y$ [So u is also a function of x, since y is]

$\Rightarrow \frac{du}{dx} = 1 - \frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = 1 - \frac{du}{dx}$ → Replace in original eqn.: $\frac{dy}{dx} = x - y$

$$1 - \frac{du}{dx} = u$$

$$1 - u = \frac{du}{dx}$$

$$\int dx = \int \frac{1}{1-u}\, du$$

$$x + C = -\ln|1-u|$$

$$-x + C = \ln|1-u|$$

$$e^{-x} \cdot e^{C} = |1-u|$$

$$\pm e^{C} \cdot e^{-x} = 1 - u$$

$$ce^{-x} = 1 - u$$

$$= 1 - (x - y)$$

$$ce^{-x} - 1 + x = y$$

Same Solution!

Remark: This Substitution Method always works for equations of the form:

$$\frac{dy}{dx} = F(ax+by+c), \quad b \neq 0.$$

Make the substitution:

$$u = ax+by+c$$

$$\frac{du}{dx} = a + b\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{b}\left(\frac{du}{dx} - a\right)$$

Replace in original eqn.:

$$\frac{1}{b}\left(\frac{du}{dx} - a\right) = F(u)$$

$$\frac{du}{dx} - a = bF(u)$$

$$\frac{du}{dx} = bF(u) + a \quad \leftarrow \text{Separable!}$$

$$\frac{1}{bF(u)+a}\,du = dx \quad \leftarrow \text{Integrate both sides.}$$

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**Ex:** $x\frac{dy}{dx} + 2y = 4x^2$ Substitution method no longer works.

$$\frac{dy}{dx} = \underbrace{-2\frac{y}{x} + 4x}_{\text{Not } F(ax+by+c)}$$

? So let's look more closely to the $\mu(x)$ method.

## First order linear ODE: $\boxed{\frac{dy}{dx} + P(x)y = g(x)}$ ← Standard Form

$\boxed{a_1(x)\frac{dy}{dx} + a_0(x)y = h(x)}$ ← General Form

↳ Can always be put in Standard Form by dividing by $a_1(x)$ [provided $a_1(x) \neq 0$]

- $g(x) = 0 \Rightarrow$ homogeneous : $\frac{dy}{dx} + P(x)y = 0$
- $g(x) \neq 0 \Rightarrow$ non-homogeneous

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## General Method of Integrating Factors:

Given: $\frac{dy}{dx} + p(x)y = g(x) \quad |\times\mu(x)$

$$\underbrace{\mu(x)\frac{dy}{dx} + p(x)\mu(x)\,y(x)}_{\text{We want this to be}} = g(x)\mu(x)$$

We want this to be $= \frac{d}{dx}\left[\mu(x)y(x)\right] = \mu(x)\frac{dy}{dx} + \frac{d\mu}{dx}y(x)$

$\Rightarrow$ We want $\frac{d\mu}{dx} = p(x)\mu(x)$

$$\mu'(x) = p(x)\mu(x)$$

$$\frac{\mu'(x)}{\mu(x)} = p(x) \Rightarrow \frac{d}{dx}\left[\ln(\mu(x))\right] = p(x)$$

$$\Rightarrow \ln\mu(x) = \int p(x)\,dx$$

$$\Rightarrow \boxed{\mu(x) = e^{\int p(x)dx}}$$

Integrating Factor

$(\ln ☺)' = \frac{☺'}{☺}$

(Chain Rule)

⇒ The equation then becomes

$$\frac{d}{dx}\left[y(x)\mu(x)\right] = \mu(x)g(x)$$

$$y(x)\mu(x) = \int \mu(x)g(x)\,dx$$

$$\Rightarrow \boxed{y(x) = \frac{1}{\mu(x)}\int \mu(x)g(x)\,dx}$$

Summary: Solving $\boxed{\frac{dy}{dx} + p(x)y = g(x)}$

1. Find the integrating factor: $\boxed{\mu(x) = e^{\int p(x)dx}}$
2. Multiply the eqn. by $\mu(x)$ and write it in the form
$$\frac{d}{dx}\left[\mu(x)y\right] = \mu(x)g(x)$$
3. Integrate to obtain
$$\mu(x)y = \int \mu(x)g(x)\,dx$$
4. Solve for y. ∎

Back to: 2 $x\frac{dy}{dx}+2y=4x^2$

- Put in Standard Form: (divide by x)

$$\frac{dy}{dx}+\frac{2}{x}y=4x \quad (*)$$

- So $p(x)=\frac{2}{x}$

$$\int p(x)dx=\int \frac{2}{x}dx=2\ln|x|+C$$

$$\Rightarrow \mu(x)=e^{2\ln|x|}=x^2 \qquad \boxed{\mu(x)=x^2}$$ Integrating Factor

- Multiply (*) by $\mu(x)$:

$$\underbrace{x^2\frac{dy}{dx}+2xy}=4x^3$$

$$\frac{d}{dx}\left[x^2y\right]=4x^3$$

$$x^2y=\int 4x^3dx$$

$$x^2y=x^4+C$$

$$\boxed{y=x^2+\frac{C}{x^2}}$$ General Solution

So far: $x\frac{dy}{dx}+2y=4x^2$

We put it in Standard Form: $\boxed{\frac{dy}{dx}+\frac{2}{x}y=4x}$ $(*)$

Then we Solved using Integrating Factors.

The $p(x)=\frac{2}{x}$ has a singularity at $x=0$! This is reflected in the solution: we got the general solution

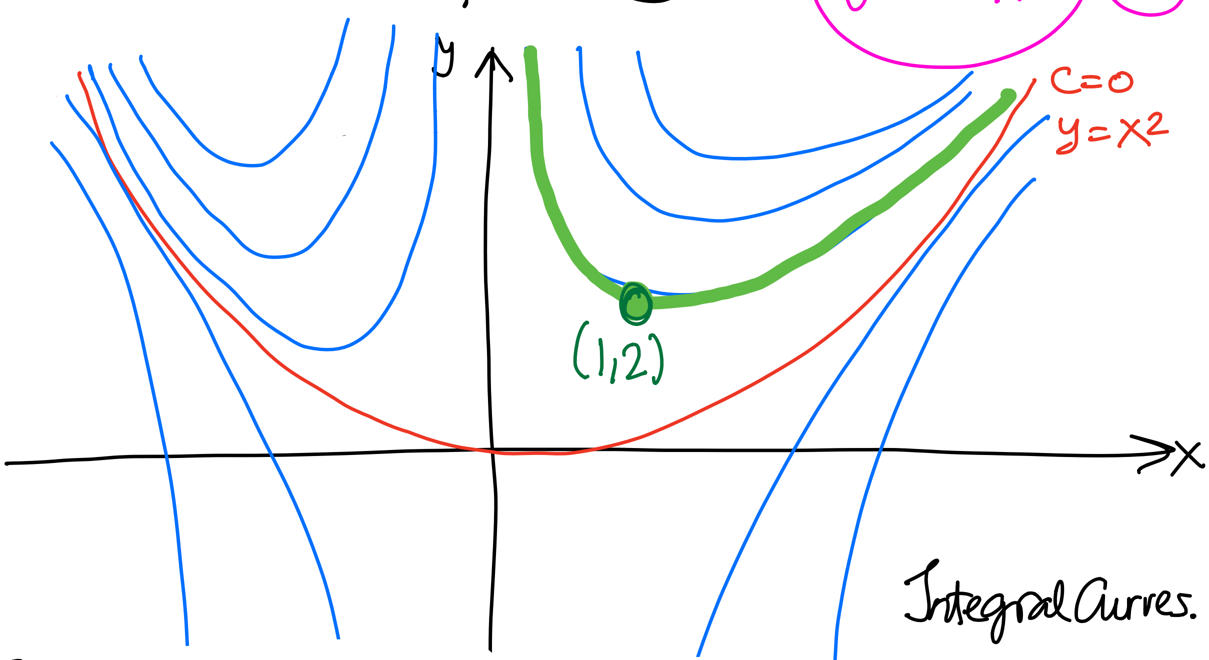
$$y=x^2+\frac{c}{x^2} \qquad (x\neq 0 \text{ when } c\neq 0)$$

This is undefined at $x=0$, UNLESS $c=0$ (in which case we get a nice function, $y=x^2$, w/no singularities).

Let's make $(*)$ an IVP: Say $y(1)=2$

$$2=1^2+\frac{c}{1^2} \Rightarrow c=1 \Rightarrow y=x^2+\frac{1}{x^2}$$

Interval: $x>0$




Integral Curves.

Remark: While $y=x^2+\frac{c}{x^2}$, $x\in(-\infty,0)\cup(0,\infty)$ for $c\neq 0$ is the general solution, the solution to the IVP is $\boxed{y=x^2+\frac{1}{x^2},\ x\in(0,\infty)}$

(b/c the point (1,2) is only contained in the $x>0$ case)

③ $x\dfrac{dy}{dx} - 4y = x^6 e^x$

Standard form: $\dfrac{dy}{dx} - \dfrac{4}{x}y = x^5 e^x \quad (**)$

$p(x) = \dfrac{-4}{x}$

$\int p(x)\,dx = -4\ln|x|$

$\mu(x) = e^{\int p(x)dx} = e^{-4\ln|x|} = |x|^{-4} = \dfrac{1}{x^4}$ $\qquad \mu(x) = \dfrac{1}{x^4}$

Multiply $(**)$ by $\mu(x)$:

$$\underbrace{\frac{1}{x^4}\frac{dy}{dx} - \frac{4}{x^5}y}_{\frac{d}{dx}\left(\frac{1}{x^4}y\right)} = xe^x$$

$$\frac{d}{dx}\left(\frac{1}{x^4}y\right) = xe^x$$

$$\Rightarrow \frac{1}{x^4}y = \int xe^x\,dx = xe^x - e^x + C$$

$$\Rightarrow \boxed{y = x^5e^x - x^4e^x + x^4C}$$

Interval of validity? $x \in (-\infty, \infty)$.

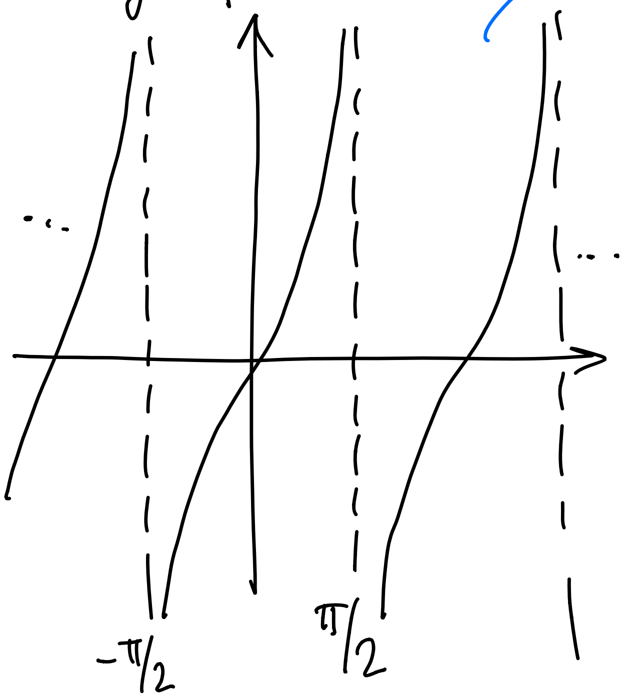
Check: $y = x^5e^x - x^4e^x + x^4C$

$\Rightarrow \dfrac{dy}{dx} = 5x^4e^x + x^5e^x - 4x^3e^x - x^4e^x + 4Cx^3$

$$\Rightarrow \underbrace{x\frac{dy}{dx} - 4y}_{\text{original eqn.}} = \left(5x^5e^x + x^6e^x - 4x^4e^x - x^5e^x + 4Cx^4\right) - 4\left(x^5e^x - x^4e^x + x^4C\right) = x^6e^x \checkmark$$

④ $y' + (\tan x)y = \cos^2 x; \quad y(0) = -1$

Tangent function:




WORK ON $x \in (-\pi/2, \pi/2)$

b/c of the initial condition.

$$p(x) = \tan x$$

$$\int p(x)dx = \int \tan(x)dx$$

$$= -\ln(\cos x) + C$$

Take $\mu(x) = e^{-\ln(\cos x)}$

$$= \frac{1}{\cos x}$$

$$\mu(x) = \frac{1}{\cos x}$$

$$\Rightarrow \boxed{\frac{1}{\cos x}y' + \frac{\sin x}{\cos^2 x}y = \cos x} \quad \text{Eqn.} \times \mu(x)$$

$$\left(\frac{1}{\cos x}y\right)' = \cos x \Rightarrow \frac{1}{\cos(x)}y = \int \cos(x)dx$$

$$= \sin(x) + C$$

$$\Rightarrow y = \sin(x)\cos(x) + C\cos(x)$$

IVP: $y(0) = -1 \Rightarrow -1 = C \Rightarrow \boxed{y = \sin(x)\cos(x) - \cos(x)}$

Interval: $x \in (-\pi/2, \pi/2)$