Last time: Integrating Factors

(Standard Form): $\frac{dy}{dx} + p(x)y = g(x)$

- Integrating factor: $\mu(x) = e^{\int p(x)dx}$
- Multiply eqn. by $\mu(x)$ and obtain: $\frac{d}{dx}[\mu(x)y] = \mu(x)g(x)$
- Integrate: $\mu(x)y = \int \mu(x)g(x)dx.$
- Solve for y!

① $y' + \frac{2}{t}y = \frac{\cos(t)}{t^2}$; $y(\pi) = 0$ → IVP.

Remark: You can already see from the equation that $t \neq 0$.
Since the initial condition is given at $t = \pi$, choose to work on $(0, \infty)$.

$p(t) = \frac{2}{t}$

$\int p(t)dt = 2\ln(t)$

$\mu(t) = e^{2\ln(t)} = (e^{\ln(t)})^2 = t^2$

$$\underbrace{t^2y' + 2ty}_{\frac{d}{dt}(t^2y)} = \cos(t)$$

$$\frac{d}{dt}(t^2y) = \cos(t) \Rightarrow t^2y = \sin(t) + C$$

$$\Rightarrow y = \frac{1}{t^2}(\sin(t) + C)$$

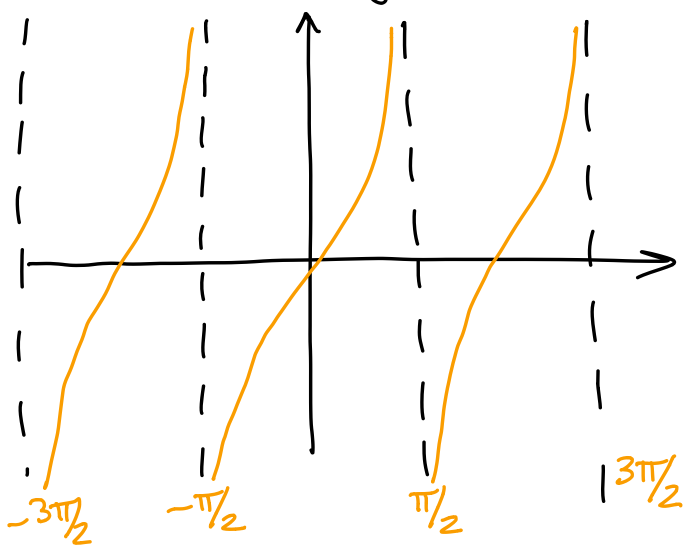
$$y(\pi) = 0 \Rightarrow 0 = \frac{1}{\pi^2}(\sin(\pi) + C)$$

$$0 = C \Rightarrow C = 0$$

$$\boxed{y = \frac{\sin(t)}{t^2} \quad ; \ t > 0}$$

② $y' + (\tan x)y = \cos^2 x$; $y(0) = -1$.

Recall the tangent function:



Because the initial condition is given at $x=0$, we choose to work on

$$x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

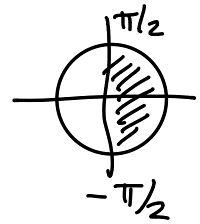
(the branch of the tangent function that contains $x=0$).

$p(x) = \tan(x)$

$\int p(x)dx = \int \tan(x)dx = -\ln|\cos x|$

$= -\ln(\cos x)$

$x \in (-\pi/2, \pi/2)$ so $\cos x > 0$!



$$\mu(x) = e^{-\ln(\cos x)} = \frac{1}{e^{\ln(\cos x)}} = \frac{1}{\cos x}$$

$$\Rightarrow \underbrace{\frac{1}{\cos x}y' + \frac{\sin x}{\cos^2 x}y}_{\frac{d}{dx}\left(\frac{1}{\cos x}y\right)} = \cos x$$

$$\frac{d}{dx}\left(\frac{1}{\cos x}y\right) = \cos x \Rightarrow \frac{1}{\cos x}y = \sin x + C$$

$$\Rightarrow y = \sin x \cos x + C \cos x$$

$y(0) = -1 \Rightarrow -1 = 0 + C \Rightarrow C = -1$

Sol.: $\boxed{y = \sin x \cos x - \cos x \;;\; x \in (-\pi/2, \pi/2)}$

(3) $x\,dy = (x\sin x - y)\,dx$ (divide by dx) $\rightarrow$ HW2 II. #5

$x y' = x\sin x - y$

$xy' + y = x\sin x$ (must divide by x now to bring to standard form).

$$\boxed{y' + \frac{1}{x} y = \sin x}$$ Standard form

$p(x) = \frac{1}{x}$; $\int p(x)\,dx = \ln|x| \Rightarrow \mu(x) = e^{\ln|x|} = |x|$

ok ✓

Take $x>0$ (make a choice; note that the wording of this HW problem says to find "an" interval where your solution is valid).

If $x>0$: $\mu(x) = |x| = x$

$$\Rightarrow \underbrace{xy' + y}_{} = x\sin x$$

$$\frac{d}{dx}(xy) = x\sin x$$

$$xy = \int x\sin x\,dx$$

$$= -x\cos x + \sin x + C$$

$$\Rightarrow \boxed{y = -\cos x + \frac{1}{x}\sin x + \frac{C}{x} \ ; \ x \in (0,\infty)}$$

④ $x(x-2)y' + 2y = 0; \quad y(3) = 6$

Must divide by $x(x-2)$ to bring to standard form

$$y' + \frac{2}{x(x-2)}\,y = 0$$

Partial Fractions:

$$\frac{2}{x(x-2)} = \frac{A}{x} + \frac{B}{x-2}$$

$$2 = A(x-2) + Bx$$

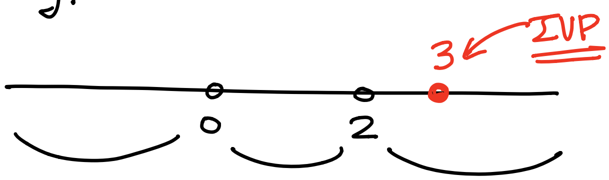
$x=2$: $2 = 2B$ $\quad B=1$

$x=0$: $2 = -2A$ $\quad A=-1$

$$p(x) = \frac{2}{x(x-2)}$$

$$= \frac{-1}{x} + \frac{1}{x-2}$$

$$\int p(x)\,dx = -\ln|x| + \ln|x-2|$$



So choose $x \in (2, \infty)$.

$$x>2 \Rightarrow \int p(x)\,dx = -\ln(x) + \ln(x-2) = \ln\left(\frac{x-2}{x}\right)$$

$$\Rightarrow \mu(x) = e^{\ln\left(\frac{x-2}{x}\right)} = \frac{x-2}{x}.$$

$$\Rightarrow \underbrace{\frac{x-2}{x}\,y' + \frac{2}{x^2}\,y}_{\frac{d}{dx}\left(\frac{x-2}{x}y\right)} = 0$$

$$\frac{d}{dx}\left(\frac{x-2}{x}\,y\right) = 0 \Rightarrow \frac{x-2}{x}\,y = C \Rightarrow y = \frac{Cx}{x-2}$$

$$y(3) = 6 \Rightarrow \frac{3C}{1} = 6 \Rightarrow C = 2$$

Solution: $\boxed{y = \dfrac{2x}{x-2};\ x \in (2, \infty)}$

⑤ $(x+2)^2 y' = 5 - 8y - 4xy$ ← Bring to Standard form

$$\underbrace{-8y-4xy}_{\text{must be brought together}}$$

$-4y(2+x)$

$$(x+2)^2 y' = 5 - 4(x+2)y$$

$$(x+2)^2 y' + 4(x+2)y = 5$$

$$\boxed{y' + \frac{4}{x+2}y = \frac{5}{(x+2)^2}}$$ Standard form

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⑥ $(x+4y^2)\,dy + 2y\,dx = 0$

→ alarm bells here that it's not linear in y

$$x\,dy + 4y^2 dy + 2y\,dx = 0$$

Try to see if it's linear in x ?? Divide by dx

$$x + 4y^2 + 2y\frac{dx}{dy} = 0$$

$$2y\frac{dx}{dy} + x = -4y^2 \quad /:2y$$

$$\boxed{\frac{dx}{dy} + \frac{1}{2y}x = -2y}$$ Linear in x !

$$p(y) = \frac{1}{2y} \Rightarrow \int p(y)\,dy = \frac{1}{2}\ln y \quad (\text{Take } y>0)$$

$$\Rightarrow \mu(y) = e^{\frac{1}{2}\ln y} = \left(e^{\ln y}\right)^{1/2} = y^{1/2} = \sqrt{y}.$$

$$\Longrightarrow \sqrt{y}\frac{dx}{dy} + \frac{1}{2\sqrt{y}}x = -2y\sqrt{y}$$

$$\frac{d}{dy}\left(\sqrt{y}\,x\right) = -2y\sqrt{y} \Rightarrow \sqrt{y}\,x = \int -2y\sqrt{y}\,dy = -2\int y^{3/2}dy$$

$$= -2\cdot\frac{2}{5}y^{5/2} + C$$

$$\Rightarrow \boxed{x = -\frac{4}{5}y^2 + \frac{C}{\sqrt{y}};\ y>0}$$