Last time:

Key concept: We had an ODE:

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

and we found a function $f(x,y)$ such that

$$M(x,y) = \frac{\partial f}{\partial x} \quad \& \quad N(x,y) = \frac{\partial f}{\partial y}$$

Then the ODE became

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0 \iff \frac{df}{dx} = 0 \iff \boxed{f(x,y) = C}$$

Implicit Sol.

This was an exact equation.

Q: When is this possible? i.e. When can we find such a function $f(x,y)$??

Given two functions $M(x,y)$, $N(x,y)$, when does there exist a function $f(x,y)$ such that

$$\frac{\partial f}{\partial x} = M; \quad \frac{\partial f}{\partial y} = N \quad ?$$

① If such f exists, then note that

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial M}{\partial y}; \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial N}{\partial x}$$

Same! (Assuming all these partial derivs. exist)

$$\implies \boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.}$$

② Let's assume now that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Does this mean f exists?

Look at $\frac{\partial f}{\partial x}(x,y) = M(x,y)$ → Integrate wrt. x (hold y constant):

$$\Rightarrow f(x,y) = \int M(x,y)\,dx$$
$$= Q(x,y) + h(y)$$

Here Q is an antiderivative of M wrt x, so you can make it something like

$$Q(x,y) = \int_{x_0}^{x} M(t,y)\,dt$$

Now, if $f(x,y) = Q(x,y) + h(y)$, then

$$\underbrace{\frac{\partial f}{\partial y}(x,y)}_{} = \frac{\partial Q}{\partial y}(x,y) + h'(y) \stackrel{?}{=} N(x,y)$$

↳ Want this to be $N(x,y)$

$$\Rightarrow h'(y) = \underbrace{N(x,y) - \frac{\partial Q}{\partial y}(x,y)}_{} \quad (*)$$

this must be a function of y only for this to work (b/c h does not depend on x)

$$\frac{d}{dx}\left(N(x,y) - \frac{\partial Q}{\partial y}(x,y)\right) = \frac{\partial N}{\partial x} - \underbrace{\frac{\partial^2 Q}{\partial x \partial y}(x,y)}_{\frac{\partial}{\partial y}\left(\underbrace{\frac{\partial Q}{\partial x}}_{= M,\text{ by construction of } Q}\right)} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

$$= \boxed{0}$$

So, since the x derivative of $\left(N - \frac{\partial Q}{\partial y}\right)$ is $\underline{\underline{0}}$, this function indeed does not depend on x.

Then in $(*)$ we can integrate wrt. y and find h. $\blacksquare$

Def.: A differential equation

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \qquad (1)$$

is called an *exact equation* provided there is a function $f(x,y)$ — called a *potential function* — such that

$$\frac{\partial f}{\partial x} = M \quad \text{and} \quad \frac{\partial f}{\partial y} = N.$$

Then, the original ODE (1) becomes simply

$$\frac{df}{dx} = 0,$$

with (implicit) solution

$$f(x,y) = c.$$

Theorem: An equation in the form of (1) is an exact equation if and only if:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

1) $2xy\,dx + (x^2-1)\,dy = 0.$

$$\left.\begin{array}{l} M = 2xy \Rightarrow \dfrac{\partial M}{\partial y} = 2x \\ N = x^2-1 \Rightarrow \dfrac{\partial M}{\partial x} = 2x \end{array}\right\} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

<u>Exact Eqn.</u>

So now: find $f(x,y)$ such that

$$\frac{\partial f}{\partial x} = 2xy \quad \& \quad \frac{\partial f}{\partial y} = x^2-1.$$

$$\boxed{\frac{\partial f}{\partial x} = 2xy} \Rightarrow \boxed{f(x,y) = x^2y + g(y)}$$

- thought process here: integrate in x (so hold y constant).
- the $g(y)$ is the "$+c$" in this situation (b/c it doesn't see x, so $\frac{d}{dx} g(y) = 0$).

So we know so far about f:

$$f(x,y) = x^2y + g(y) \Rightarrow \frac{\partial f}{\partial y} = x^2 + g'(y)$$

Want: $\dfrac{\partial f}{\partial y} = x^2 - 1.$

$$-1 = g'(y)$$
$$-y = g(y)$$

$\Rightarrow f(x,y) = x^2y - y \Rightarrow$ Solution to ODE:

<u>Potential</u>

$$\boxed{x^2y - y = c} \Rightarrow y = \frac{c}{x^2-1}$$

2 $\left(e^{2y}-y\cos(xy)\right)dx+\left(2xe^{2y}-x\cos(xy)+2y\right)dy=0.$

Check: Exact?

$$M=e^{2y}-y\cos(xy) \Rightarrow \frac{\partial M}{\partial y}=2e^{2y}-\cos(xy)+xy\sin(xy)$$

$$N=2xe^{2y}-x\cos(xy)+2y \Rightarrow \frac{\partial N}{\partial x}=2e^{2y}-\cos(xy)+xy\sin(xy)$$

Yes, this is an exact eqn.

Find the potential [need $\frac{\partial f}{\partial x}=M$, $\frac{\partial f}{\partial y}=N$].

$$\frac{\partial f}{\partial x}=e^{2y}-y\cos(xy) \Rightarrow f(x,y)=xe^{2y}-\sin(xy)+g(y)$$

$$\Downarrow$$

$$\frac{\partial f}{\partial y}=2xe^{2y}-x\cos(xy)+g'(y)$$

$$N=2xe^{2y}-x\cos(xy)+2y$$

$$\Rightarrow g'(y)=2y \Rightarrow g(y)=y^2$$

$$\Rightarrow f(x,y)=xe^{2y}-\sin(xy)+y^2$$

$\Rightarrow$ Solutions to ODE: $\boxed{xe^{2y}-\sin(xy)+y^2=c}$

3. $\underbrace{(\cos x \sin x - xy^2)}_{M}\,dx + \underbrace{y(1-x^2)}_{N}\,dy = 0 \;;\; y(0)=2$

IVP

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= -2xy \\ \frac{\partial N}{\partial x} &= y(-2x) = -2xy \end{aligned}\right\} \checkmark \text{ Exact Eqn.}$$

Potential:

$$\frac{\partial f}{\partial x} = \cos(x)\sin(x) - xy^2 \quad (M)$$

$$\int \cos(x)\sin(x)\,dx = \frac{1}{2}\sin^2(x)$$

u-sub, $u = \sin(x)$, $du = \cos(x)\,dx$

$$\Rightarrow f = \frac{1}{2}\sin^2(x) - \frac{x^2}{2}y^2 + g(y)$$

$$\Downarrow$$

$$\left.\begin{aligned} \frac{\partial f}{\partial y} &= -\frac{x^2}{2}\cdot 2y + g'(y) = -x^2y + g'(y) \\ N &= -x^2y + y \end{aligned}\right\} \Rightarrow \begin{aligned} g'(y) &= y \\ g(y) &= \frac{y^2}{2} \end{aligned}$$

$$\Rightarrow \frac{1}{2}\sin^2(x) - \frac{x^2y^2}{2} + \frac{y^2}{2} = C$$

(Same:) $\boxed{\sin^2(x) - x^2y^2 + y^2 = C}$ General Sol.

IVP: $y(0)=2 \Rightarrow \sin^2(0) - 0^2\cdot 2^2 + 2^2 = C \Rightarrow C=4$

$\Rightarrow$ Sol.: $\boxed{\sin^2(x) - x^2y^2 + y^2 = 4}$ Implicit Sol. (*)

NOTE: When working the $\int \sin(x)\cos(x)$ integral, we could have used a different substitution:

$$\int \sin(x)\cos(x)\,dx = \frac{-1}{2}\cos^2(x).$$

$u = \cos(x)$

$du = -\sin(x)\,dx$

Then $\frac{\partial f}{\partial x} = \cos(x)\sin(x) - xy^2$ would yield

$$f = \frac{-1}{2}\cos^2(x) - \frac{x^2}{2}y^2 + g(y)$$

$$\Rightarrow \left.\begin{aligned} \frac{\partial f}{\partial y} &= -x^2y + g'(y) \\ N &= -x^2y + y \end{aligned}\right\} \Rightarrow g'(y) = \frac{y^2}{2}$$

The general sol. would then be:

$$\frac{-1}{2}\cos^2(x) - \frac{x^2y^2}{2} + \frac{y^2}{2} = C$$

or

$$\boxed{-\cos^2(x) - x^2y^2 + y^2 = C} \quad (**)$$

This is <u>the same</u> as the first gen. sol.:

$$\boxed{\sin^2(x) - x^2y^2 + y^2 = C}$$

b/c $-\cos^2(x) = \sin^2(x) - 1$ and C can absorb the (-1).

Plug the IVP condition in $(**)$: $y(0) = 2$

$$-\cos^2(0) - 0^2 \cdot 2^2 + 2^2 = C \Rightarrow C = 3$$

$$\Rightarrow \quad \boxed{-\cos^2(x) - x^2y^2 + y^2 = 3}$$

$-\cos^2(x) = \sin^2(x) - 1$

$(\sin^2 x - 1) - xy^2 + y^2 = 3$

$\sin^2 x - xy^2 + y^2 = 4.$

This is also the same as

$$\boxed{\sin^2(x) - x^2y^2 + y^2 = 4}$$

Implicit Sol. $(\cancel{x})$

Another note: You don't have to start with $\frac{\partial f}{\partial x}$.

$$\underbrace{(\cos x \sin x - xy^2)}_{M} dx + \underbrace{y(1-x^2)}_{N} dy = 0$$

$$\underbrace{\frac{\partial f}{\partial y} = y(1-x^2)}_{\text{Integrate } dy} \Rightarrow f = \frac{y^2}{2}(1-x^2) + g(x)$$

$$\Downarrow$$

$$\frac{\partial f}{\partial x} = \frac{y^2}{2}(-2x) + g'(x) = -xy^2 + g'(x)$$

$$M = \cos(x)\sin(x) - xy^2$$

$$\Rightarrow g'(x) = \cos(x)\sin(x) \Rightarrow g = \frac{1}{2}\sin^2(x)$$

$$\Rightarrow f = \frac{y^2}{2}(1-x^2) + \frac{1}{2}\sin^2(x)$$

Same as before.

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$$(x+y)\,dx + x\ln x\,dy = 0 \qquad \Big|\cdot \frac{1}{x}$$

$$\underbrace{(x+y)}_{M} \qquad \underbrace{x\ln x}_{N}$$

$$\left.\begin{aligned}\frac{\partial M}{\partial y} &= 1\\ \frac{\partial N}{\partial x} &= \ln x + 1\end{aligned}\right\} \Rightarrow \text{NOT exact}$$

ODE $\cdot \frac{1}{x}$:

$$\underbrace{\left(1+\frac{y}{x}\right)}_{M_1}dx + \underbrace{(\ln x)}_{N_1}dy = 0$$

$$\left.\begin{aligned}\frac{\partial M_1}{\partial y} &= \frac{1}{x}\\ \frac{\partial N_1}{\partial x} &= \frac{1}{x}\end{aligned}\right\} \text{Exact!}$$

$f(x,y)$ such that:

$$\frac{\partial f}{\partial x} = 1 + \frac{y}{x}$$

$$\frac{\partial f}{\partial y} = \ln x \quad \Rightarrow \quad f(x,y) = y\ln x + g(x)$$

$$\left.\begin{aligned}\frac{\partial f}{\partial x} &= \frac{y}{x} + g'(x)\\ &= \frac{y}{x} + 1\end{aligned}\right\} \Rightarrow \begin{aligned} g'(x) &= 1\\ g(x) &= x\end{aligned}$$

$$\Rightarrow f(x,y) = y\ln x + x$$

$$\boxed{y\ln x + x = C} \quad \text{Solution}$$

Exact Eqns. and Integrating Factors:

Sometimes an equation of the form:

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

which is NOT exact ($M_y \neq N_x$) can be "turned" into an exact equation:

If there exists a function $\mu(x,y)$ - called an integrating factor - such that:

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x} \qquad (*)$$

then multiplying the original eqn. by $\mu(x)$ yields an exact eqn.

Problem: Solving $(*)$ is often times very difficult.

$\Rightarrow$ Integrating factors are a powerful tool for exact eqns., but unfortunately can only be found in particular cases.

5 $\underbrace{y(x+y+1)}_{M}\,dx + \underbrace{(x+2y)}_{N}\,dy = 0$; $\mu(x,y) = e^x$

$$\left.\begin{aligned} M_y &= (x+y+1) + y = x+2y+1 \\ N_x &= 1 \end{aligned}\right\}$$ Not exact (original ODE)

ODE $\cdot\, e^x$:
$$\underbrace{ye^x(x+y+1)}_{M_1}\,dx + \underbrace{(x+2y)e^x}_{N_1}\,dy = 0$$

$$\left.\begin{aligned} \frac{\partial M_1}{\partial y} &= e^x(x+y+1) + ye^x = xe^x + 2ye^x + e^x \\ \frac{\partial N_1}{\partial x} &= e^x + (x+2y)e^x = xe^x + 2ye^x + e^x \end{aligned}\right\}$$ ✓ Exact

$\Rightarrow$ find f such that

$$\frac{\partial f}{\partial x} = ye^x(x+y+1) = yxe^x + y^2e^x + ye^x$$

$$\frac{\partial f}{\partial y} = (x+2y)e^x = xe^x + 2ye^x$$

$$\int xe^x\,dx$$

$u = x \quad dv = e^x dx$

$du = dx \quad v = e^x$

$$= xe^x - \int e^x dx$$

$$= xe^x - e^x$$

$$f = \int (yxe^x + y^2e^x + ye^x)\,dx$$

$$= y(xe^x - e^x) + y^2e^x + ye^x + g(y)$$

$$\Rightarrow \frac{\partial f}{\partial y} = xe^x - \cancel{e^x} + 2ye^x + \cancel{e^x} + g'(y)$$

$$= xe^x + 2ye^x + g'(y)$$

$$= xe^x + 2ye^x$$

$\Rightarrow g'(y) = 0 \Rightarrow$ Take $g = 0$

$$\Rightarrow f = xye^x - \cancel{ye^x} + y^2e^x + \cancel{ye^x}$$

$$\boxed{xye^x + y^2e^x = C}$$

Check:

$$ye^x + \underbrace{xy'e^x} + xye^x + \underbrace{2yy'e^x} + y^2e^x = 0$$

$$\left(ye^x + xye^x + y^2e^x\right) + \left(xe^x + 2ye^x\right)y' = 0$$

$$\left(y + xy + y^2\right) + (x+2y)\,y' = 0$$

$$\boxed{y(1+x+y) + (x+2y)\,y' = 0}$$