Chapter 3: Second-Order Linear ODEs.

Our main concern in this chapter:

$$\boxed{\begin{gathered} a_2(x)\,y'' + a_1(x)\,y' + a_0(x)\,y = g(x) \\ \text{IVP: } \quad y(x_0) = y_0;\; y'(x_0) = y_1 \end{gathered}} \qquad (1)$$

We'll be looking for solutions to (1) on a specific interval I.
This is because of the following VIP theorem (whose proof is typically done in a 400-level, proof-based ODE course):

Existence & Uniqueness Theorem:

Suppose we are working on an open interval I where:

- $a_2(x), a_1(x), a_0(x)$ & $g(x)$ are continuous on I.
- $a_2(x) \neq 0$ for all $x \in I$

Then, if $x_0 \in I$ is any point in I, then a solution to the IVP (1) exists on the interval I and it is unique.

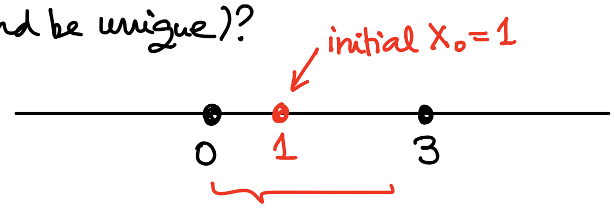
Example: $y'' + xy' + x^2 y = 0$
$y(1) = 0;\; y'(1) = 0$

$y \equiv 0$ is a solution to this IVP, so it is the only solution on any interval containing $x_0 = 1$.

Example: $(x^2 - 3x)\,y'' + xy' - (x+3)\,y = 0$
$y(1) = 2;\; y'(1) = 1.$

Q: What is the largest possible interval in which a solution is guaranteed to exist (and be unique)?

need: $x^2 - 3x \neq 0$
$x(x-3) \neq 0$
$x \neq 0, 3$



Interval: $(0, 3)$.

Remark: Sometimes the initial conditions are given at different values of x:

$$a_2(x)y''+a_1(x)y'+a_0(x)y=g(x)$$

$$y(x_0)=y_0\,;\ y'(x_1)=y_1$$

If $x_0\neq x_1$, this is called a <u>boundary value problem</u> (BVP) (not so much in this class; behave not as nicely as IVP's).

Remark : Recall that in (1),

- if $g(x)\equiv 0$, the eqn. is called <u>homogeneous</u>
- if $g(x)\not\equiv 0$, the eqn. is called <u>nonhomogeneous</u>.

For now: we focus on <u>homogeneous</u> equations:

$$\boxed{a_2(x)y''+a_1(x)y'+a_0(x)y=0} \qquad (2)$$

Principle of Superposition:

Suppose $y_1, y_2, \ldots, y_k$ are all solutions to the homogeneous linear eqn. (2) on an interval I. Then ANY <u>linear combination</u>:

$$y=c_1y_1(x)+c_2y_2(x)+\ldots+c_ky_k(x)$$

(where $c_1,\ldots,c_k$ are arbitrary constants), is also a solution to (2) on the interval I.

Quick proof for the case $n=k=2$:

Assume that y_1, y_2 are both solutions to the equation

$$a_2(x)y''+a_1(x)y'+a_0(x)y=0.$$

Let c_1, c_2 be any two real numbers and let

$$Y(x) := c_1 y_1(x) + c_2 y_2(x)$$

$$\Rightarrow \quad Y' = c_1 y_1' + c_2 y_2'$$

$$Y'' = c_1 y_1'' + c_2 y_2''$$

$$\Rightarrow a_2 Y'' + a_1 Y' + a_0 Y = a_2\left(c_1 y_1'' + c_2 y_2''\right) + a_1\left(c_1 y_1' + c_1 y_2'\right) + a_0\left(c_1 y_1 + c_2 y_2\right)$$

$$= \left(a_2 c_1 y_1'' + a_1 c_1 y_1' + a_0 c_1 y_1\right) + \left(a_2 c_2 y_2'' + a_1 c_1 y_2' + a_0 c_2 y_2\right)$$

$$= c_1 \underbrace{\left(a_2 y_1'' + a_1 y_1' + a_0 y_1\right)}_{=0 \text{ b/c } y_1 \text{ is a solution to } (2)} + c_2 \underbrace{\left(a_2 y_2'' + a_1 y_2' + a_0 y_2\right)}_{=0 \text{ b/c } y_2 \text{ is a solution to } (2)}$$

$$= \underline{\underline{0}}$$

$\Rightarrow Y = c_1 y_1 + c_2 y_2$ is indeed also a solution to (2). ∎

2 immediate consequences:

(1) If $y_1(x)$ is a solution to a homogeneous linear ODE (2), then $c_1 y_1(x)$ is <u>also a solution</u>, for any constant c_1.

(2) A homogeneous linear eqn. <u>always</u> possesses the <u>trivial solution</u> $y \equiv 0$:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = 0$$

$y \equiv 0$ always satisfies this.

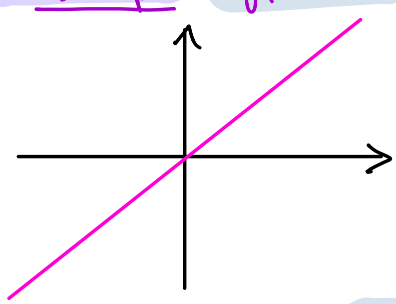
We will explore this in more detail next class, but for now:

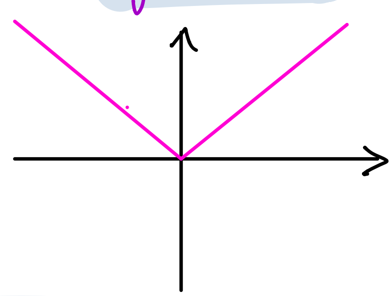
- Two functions $y_1(x), y_2(x)$ are **linearly dependent** <u>on an interval I</u> if one is a constant multiple of each other, i.e. there is a constant c such that

$$y_1(x) = c \cdot y_2(x), \text{ for } \underline{\text{all}}\ x \in I.$$

Otherwise, they are called **linearly independent.**
(a more general definition exists here - next time).

<u>Example</u>: $y_1(x) = x$ $\qquad$ $y_2(x) = |x|$





- y_1, y_2 are linearly independent <u>on $\mathbb{R}$</u>
- BUT, <u>on $(0,\infty)$</u>, they are linearly <u>dependent</u>:

$$y_1(x) = y_2(x), \text{ for all } x \in (0,\infty).$$

Also, <u>on $(-\infty, 0)$</u>, they are linearly <u>dependent</u>:

$$y_1(x) = -y_2(x), \text{ for all } x \in (-\infty, 0).$$

A pair of functions $y_1(x), y_2(x)$ which are:

- Solutions to a homogeneous linear 2^{nd} order ODE

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

on an interval I

- and are linearly independent on I

is called a fundamental set of solutions on the interval I,

Theorem:

Suppose $\{y_1, y_2\}$ is a fundamental set of solutions to the homogeneous linear ODE:

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

Then ANY solution y to this ODE will be of the form:

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

for some constants c_1, c_2.

we will go over this in detail next lecture. more likely next Thursday

Point for now: If you have a homogeneous 2^{nd} order ODE and you find two (linearly indp.) solutions y_1, y_2 then the general solution will be simply

$$y = c_1 y_1 + c_2 y_2.$$

Homogeneous Linear 2^{nd} order eqns. with constant coefficients:

$$\boxed{ay'' + by' + cy = 0} \qquad (3)$$

(where a, b, c are constants; $a \neq 0$)

Surprising fact: all solutions of equations of the form (3) will end up being exponentials, or more explicitly linear combinations of exponentials.

Let's try to find a solution to (3) in the form $y(x) = e^{mx}$:

$$y = e^{mx}$$
$$y' = me^{mx}$$
$$y'' = m^2 e^{mx}$$

So for our y to satisfy (3), we would need:

$$a \cdot m^2 e^{mx} + b \cdot me^{mx} + ce^{mx} = 0$$

$$(am^2 + bm + c)\underbrace{e^{mx}}_{\text{can never be } 0} = 0$$

$$\Rightarrow \quad \boxed{am^2 + bm + c = 0}$$

The Characteristic Equation of (3).

We'll have 3 situations here:
- 2 distinct real roots
- 1 real (repeated) root
- 2 complex roots

Case 1: 2 distinct real roots.

If the characteristic equation has two distinct real roots, m_1 and m_2, then we have that

$$y_1 = e^{m_1 x} \text{ and } y_2 = e^{m_2 x}$$

is a fundamental set of solutions for (3) [clearly linearly indp. b/c $m_1 \neq m_2$], so then our general solution is simply:

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

① $2y'' - 5y' - 3y = 0$

Char. Eqn.: $2m^2 - 5m - 3 = 0$

$$\Delta = 25 - 4 \cdot 2 \cdot (-3) = 25 + 24 = 49 > 0$$

$$m_{1,2} = \frac{5 \pm \sqrt{49}}{4} = \frac{5 \pm 7}{4} = 3, -\tfrac{1}{2}$$

Solutions: $m_1 = 3$; $m_2 = -\frac{1}{2}$ (2 distinct real roots)

$\Rightarrow$ gen. sol.: $\boxed{y = c_1 e^{3x} + c_2 e^{-\frac{1}{2}x}}$

$y' = 3c_1 e^{3x} - \frac{1}{2} c_2 e^{-\frac{1}{2}x}$

What if it was an IVP?

$$\begin{cases} y(0) = 1 \\ y'(0) = 2 \end{cases} \Rightarrow \begin{cases} c_1 + c_2 = 1 \\ 3c_1 - \frac{1}{2} c_2 = 2 \end{cases}$$

$c_1 + c_2 = 1$

$6c_1 - c_2 = 4$

$\oplus$ $7c_1 = 5$

$c_1 = 5/7$

$\Rightarrow c_2 = 2/7$

$\boxed{y = \frac{5}{7} e^{3x} + \frac{2}{7} e^{-\frac{1}{2}x}}$ $\boxed{x \in \mathbb{R}}$

Looking ahead at the next case, when we have only one ($m_1 = m_2$) (repeated) root to the char. eqn., the problem will be that we then only have one solution: the char. eqn. only produces $y_1(x) = e^{m_1 x}$. But we need two (linearly indp.) solutions to form a fundamental set, and our general solution.

Reduction of order: Producing a second solution from a known solution.

Look at the example: $\boxed{y'' - y = 0}$ (4)

We can guess a solution to this without much effort:

$$y(x) = e^x$$

is a solution to (4). (function equal to its own 2nd derivative?)

How can we produce a second solution?

Try: $y = u(x)e^x$ where u is some function (tbd?)

$$y' = u'e^x + ue^x = (u' + u)e^x$$

$$y'' = (u'' + u')e^x + (u' + u)e^x = (u'' + 2u' + u)e^x$$

$\Rightarrow y'' - y = 0$ becomes:

$$(u'' + 2u' + u)e^x - (ue^x) = 0$$

$$(u'' + 2u')\underbrace{e^x}_{\text{can never be } 0} = 0$$

$\Rightarrow \boxed{u'' + 2u' = 0}$ $\longrightarrow$ This can be "reduced" to a first order ODE by making the substitution

$$v := u'$$

$\Rightarrow u'' + 2u' = 0$ becomes $\boxed{v' + 2v = 0}$ Separable!

$$\frac{dv}{dx} = -2v$$

$$\int \frac{-1}{2v}\,dv = \int dx$$

$$-\frac{1}{2}\ln|v| = x + C$$

$$\ln|v| = -2x + C$$

$$v = e^{-2x+C} \Rightarrow \boxed{v = c_1 e^{-2x}}$$

$$\Rightarrow u' = c_1 e^{-2x}$$

$$\Rightarrow \boxed{u = -\frac{1}{2}c_1 e^{-2x} + c_2}$$

$\Rightarrow$ we looked for a solution of the form $y = u(x)e^x$, and found that $u(x)$ qualifies as long as u is of the form:

$$u(x) = c_1 e^{-2x} + c_2$$

So pick some nice constants, $c_1 = 1, c_2 = 0$: $u(x) = e^{-2x}$

and that gets us a new solution: $\boxed{y = e^{-x}}$ $\left(u \cdot e^x = e^{-2x} \cdot e^x = e^{-x}\right)$

$\Rightarrow$ now we have two (obviously) linearly indp. solutions,

e^x & e^{-x}

$\Rightarrow$ we have our general solution: $\boxed{y = c_1 e^x + c_2 e^{-x}}$