Last time: Second order linear ODEs: (homogeneous for now)

$$a_2(x)\,y'' + a_1(x)\,y' + a_0(x)\,y = 0$$

We saw: If we have two linearly independent solutions y_1 and y_2 to a second order linear homogeneous ODE, then any other solution will be a linear combination of y_1 & y_2. In other words, the general solution is

$$\boxed{y = c_1 y_1 + c_2 y_2}$$

First focus on the case of constant coefficients:

$$(1) \quad \boxed{ay'' + by' + cy = 0} \qquad (a, b, c \in \mathbb{R};\ a \neq 0)$$

We looked for solutions of the form $y = e^{mx}$

$$\Rightarrow \quad y' = me^{mx}$$
$$y'' = m^2 e^{mx}$$

Plugging back into eqn. (1):

$$(am^2 + bm + c)\,e^{mx} = 0$$

$$\boxed{am^2 + bm + c = 0} \leftarrow \text{The characteristic equation:}$$

$$\boxed{\Delta = b^2 - 4ac}$$

A quadratic equation can have:

$\Delta > 0$ $\rightsquigarrow$ 2 distinct real solutions $m_1 \neq m_2 \in \mathbb{R}$
(last time: the general solution is then just $y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$)

today:

$\Delta = 0$ $\rightsquigarrow$ 1 repeated real root $m_1 = m_2 \in \mathbb{R}$

$\Delta < 0$ $\rightsquigarrow$ 2 complex roots $m_1, m_2 \in \mathbb{C}$. (complex conjugates).

I. The case of repeated real roots $m_1 = m_2$ ($\Delta = 0$)

The issue here is that solving the char. eqn. only gives us one solution $e^{m_1 x}$, and we need two (indp.) solutions to form the general solution.

Example:

$$\boxed{y'' + 4y' + 4y = 0} \quad (2)$$

Char. Eqn.: $m^2 + 4m + 4 = 0$

$(m+2)^2 = 0 \Rightarrow m_1 = m_2 = -2$

$\Rightarrow$ This only produces one solution, $y_1 = e^{-2x}$.

We need to find another (indp. solution):

"Reduction of order" (producing a new solution from an old one).

Main idea (Jean d'Alembert, 18th century): We know that if y_1 is a solution, then so is $c\,y_1(x)$ for any constant c. BUT: we need a linearly independent solution! So, try: instead of a constant c, let's look for a solution of the form $u(x) \cdot y_1(x)$.

So: let's look for a solution of the form $y_2(x) = u(x)\,e^{-2x}$ to (2).

$$y_2 = u e^{-2x}$$

$$y_2' = u' e^{-2x} - 2u e^{-2x} = (u' - 2u) e^{-2x}$$

$$y_2'' = (u'' - 2u') e^{-2x} - 2(u' - 2u) e^{-2x}$$

$$= (u'' - 4u' + 4u) e^{-2x}$$

Substitute these for y, y', y'' in (2):

$$(u'' - 4u' + 4u) e^{-2x} + 4(u' - 2u) e^{-2x} + 4u e^{-2x} = 0$$

$$(u'' - \cancel{4u'} + \cancel{4u} + \cancel{4u'} - \cancel{8u} + \cancel{4u}) e^{-2x} = 0$$

$$u'' \underbrace{e^{-2x}}_{\text{can never be } 0} = 0 \quad \Rightarrow u'' = 0 \Rightarrow u' = C_1 \Rightarrow u = C_1 x + C_2$$

Take $c_1=1$, $c_2=0$ (we only need some function which works)

$\Rightarrow$ take $u(x)=x$ $\Rightarrow$ $\boxed{y_2(x)=xe^{-2x}}$

(Clearly linearly independent from $y_1=e^{-2x}$)

$\Rightarrow$ General solution:

$$\boxed{y=c_1e^{-2x}+c_2xe^{-2x}}$$

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Generally: Suppose we have

$$ay''+by'+cy=0 \quad \text{with} \quad \Delta=b^2-4ac=0$$

Then the quadratic equation (char. eqn.) $am^2+bm+c=0$ has one real repeated root:

$$m_1=m_2=\frac{-b}{2a}$$

This gives us one family of solutions: $y_1=e^{-\frac{bx}{2a}}$

Same procedure as before: look for a solution

$$y_2(x)=u(x)e^{-\frac{bx}{2a}}$$

$$\Rightarrow y_2'=u'e^{-\frac{bx}{2a}}-u\frac{b}{2a}e^{-bx/2a}$$

$$=\left(u'-\frac{b}{2a}u\right)e^{-bx/2a}$$

$$\Rightarrow y_2''=\left(u''-\frac{b}{2a}u'\right)e^{-bx/2a}-\frac{b}{2a}\left(u'-\frac{b}{2a}u\right)e^{-bx/2a}$$

$$=\left(u''-\frac{\cancel{2}b}{\cancel{2}a}u'+\frac{b^2}{4a^2}u\right)e^{-bx/2a}$$

Replace y_2'', y_2', y_2 in the original eqn.:

$$a\left(u''-\frac{b}{a}u'+\frac{b^2}{4a^2}u\right)e^{-bx/2a}+b\left(u'-\frac{b}{2a}u\right)e^{-bx/2a}$$
$$+cue^{-bx/2a}=0$$

$$\underbrace{\left(au'' - \cancel{bu'} + \frac{b^2}{4a}u + \cancel{bu'} - \frac{b^2}{2a}u + cu\right)}_{=0} e^{-bx/2a} = 0$$

$$au'' + \left(\frac{b^2}{4a} - \frac{b^2}{2a} + c\right)u = 0$$

$$au'' + \left(-\frac{b^2}{4a} + c\right)u = 0$$

$$au'' + \left(-\frac{b^2 - 4ac}{4a}\right)u = 0$$

this is Δ, which is 0!

$\Rightarrow au'' = 0 \Rightarrow u'' = 0$

$u' = c_1$

$u = c_1 x + c_2$

$\Rightarrow$ As before, we have the new solution $y_2 = xe^{-bx/2a}$

$\Rightarrow$ In case of a repeated root $m_1 = m_2$, the general solution is:

$$\boxed{y = c_1 e^{m_1 x} + c_2 x e^{m_2 x}}$$

II. The case of two complex roots ($\Delta<0$)

Complex Numbers ($\mathbb{C}$):

$$\boxed{i=\sqrt{-1}}$$

$i^2=-1$
$i^3=-i$
$i^4=1$

General complex number:

$$\boxed{z=a+bi}$$

a: real part, bi: imaginary part

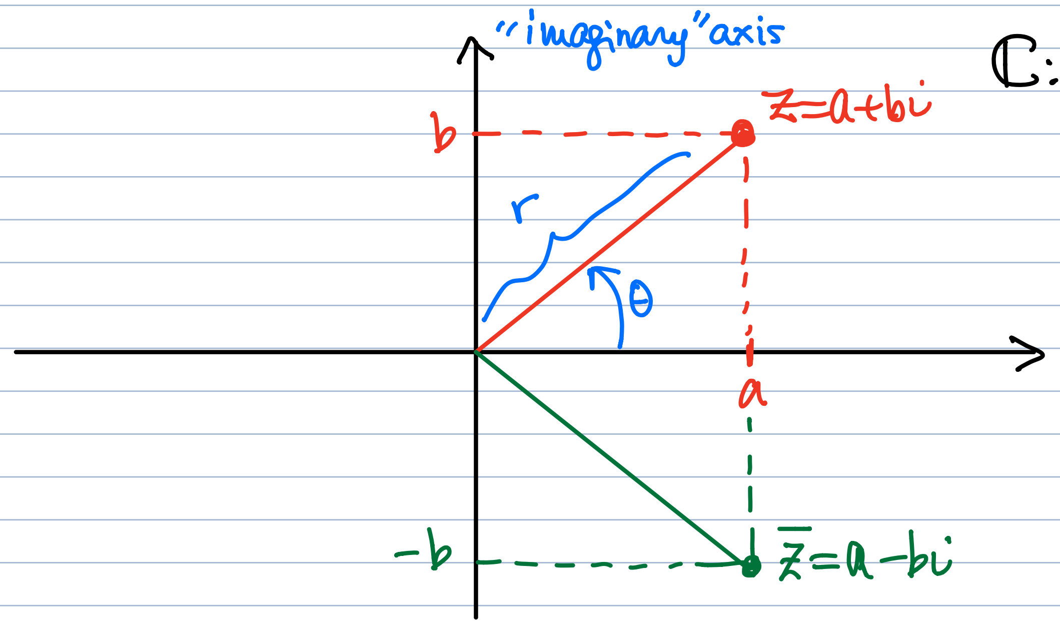
$a, b \in \mathbb{R}$

$|z|=\sqrt{a^2+b^2}=|\bar{z}|$

$r=|z|$

The complex conjugate:

$$\boxed{\bar{z}=a-bi}$$



$\mathbb{C}$: the complex plane

$r:=|z|=\sqrt{a^2+b^2} \Rightarrow$ $z=r\cos\theta+ir\sin\theta$

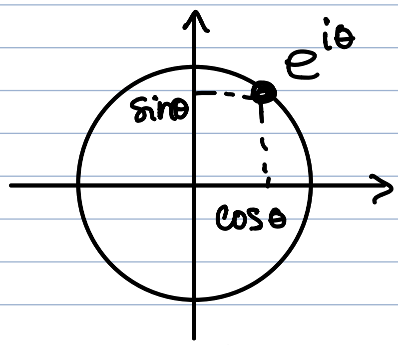
$a=r\cos\theta$
$b=r\sin\theta$

$z=r\underbrace{(\cos\theta+i\sin\theta)}_{e^{i\theta}}$

Euler's Formula!

$$e^{i\theta}=\cos\theta+i\sin\theta$$

⇒ Can think of $e^{i\theta}$ as tracing out the unit circle as θ varies.



This comes from Maclaurin series:

$$e^{x} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \frac{x}{1} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

Make x into $i \cdot x$ here:

$$e^{ix} = 1 + \frac{\overset{i}{i}x}{1} + \frac{\overset{-1}{i^2}x^2}{2!} + \frac{\overset{-i}{i^3}x^3}{3!} + \frac{\overset{1}{i^4}x^4}{4!} + \frac{\overset{i}{i^5}x^5}{5!} + \frac{\overset{-1}{i^6}x^6}{6!} + \overset{\ldots}{\ldots}$$

$$= 1 + i\frac{x}{1} - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} - i\frac{x^7}{7!} + \frac{x^8}{8!} + \ldots$$

$$= (\cos x) + i(\sin x)$$

!

In order to even define such a thing as an exponential function w/ complex exponent, one must appeal to the polynomial representation of the exponential:

$$e^{x} = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow e^{z},\ z \in \mathbb{C} \text{ is \underline{defined} as: } \boxed{e^{z} := \sum_{n=0}^{\infty} \frac{z^n}{n!}}$$

Back to our characteristic equation

$$am^2+bm+c=0; \qquad \Delta=b^2-4ac<0$$

The solutions then are <u>complex conjugates</u>

$$m_{1,2}=\frac{-b\pm\sqrt{\Delta}}{2a}=\alpha\pm i\beta$$

Our general solution is then

$$y=c_1e^{(\alpha+i\beta)x}+c_2e^{(\alpha-i\beta)x} \longrightarrow \text{Not easy to work with!}$$

Easier formula: we have 2 linearly indp. solutions:

$$y_1=e^{(\alpha+i\beta)x}=e^{\alpha x}\cdot e^{i\beta x}=e^{\alpha x}\left(\cos(\beta x)+i\sin(\beta x)\right)$$
$$y_2=e^{(\alpha-i\beta)x}=e^{\alpha x}\cdot e^{-i\beta x}=e^{\alpha x}\left(\cos(\beta x)-i\sin(\beta x)\right)$$

If these are solutions, then any linear combination of y_1 & y_2 is also a solution:

$$\tilde{y}_1:=y_1+y_2=2e^{\alpha x}\cos(\beta x)$$
$$\tilde{y}_2:=y_1-y_2=2ie^{\alpha x}\sin(\beta x)$$

(only take real solutions for now in this class)

$$\boxed{y=c_1e^{\alpha x}\cos(\beta x)+c_2e^{\alpha x}\sin(\beta x)}$$

<u>Example:</u> $y''+y'+y=0$

Char. eqn.: $m^2+m+1=0$

$\Delta=1-4=-3 \Rightarrow m_{1,2}=\dfrac{-1\pm\sqrt{-3}}{2}=\dfrac{-1\pm i\sqrt{3}}{2}$

Complex roots: $m_1=\dfrac{-1}{2}+i\dfrac{\sqrt{3}}{2}$; $m_2=\dfrac{-1}{2}-i\dfrac{\sqrt{3}}{2}$

$\Rightarrow$ General Solution: $y=c_1e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right)+c_2e^{-x/2}\sin\left(\frac{\sqrt{3}}{2}x\right)$

Summary of: $\boxed{ay''+by'+c=0}$

Characteristic Eqn.: $am^2+bm+c=0$

$\Delta=b^2-4ac$

① 2 distinct real roots $m_1 \neq m_2$ $(\Delta>0)$:

$$y=c_1e^{m_1x}+c_2e^{m_2x}$$

② 1 repeated real root $m_1=m_2$ $(\Delta=0)$:

$$y=c_1e^{m_1x}+c_2xe^{m_1x}$$

③ 2 complex conjugate roots $\alpha \pm i\beta$ $(\Delta<0)$:

$$y=c_1e^{\alpha x}\cos(\beta x)+c_2e^{\alpha x}\sin(\beta x).$$