3.5 Nonhomogeneous Eqns.: Method of Undetermined Coefficients

Nonhomogeneous second-order linear ODE's:

(NH): $y'' + p(x)y' + q(x)y = g(x)$

(p, q, g are continuous functions on some interval I).

Associated with any (NH) equation is its homogeneous version:

(H): $y'' + p(x)y' + q(x)y = 0.$

Suppose $Y_1(x)$ and $Y_2(x)$ are solutions to (NH):

$$Y_1'' + p(x)Y_1' + q(x)Y_1 = g(x)$$
$$Y_2'' + p(x)Y_2' + q(x)Y_2 = g(x)$$

$$\ominus \quad (Y_1 - Y_2)'' + p(x)(Y_1 - Y_2)' + q(x)(Y_1 - Y_2) = 0$$

$\Rightarrow (Y_1 - Y_2)$ is a solution to (H).

Lemma 1: If Y_1 and Y_2 are solutions to (NH), then $(Y_1 - Y_2)$ is a solution to (H).

$\Downarrow$

Lemma 2: If Y_1 and Y_2 are solutions to (NH), and y_1, y_2 are linearly indp. solutions to (H), then

$$Y_1 - Y_2 = c_1 y_1 + c_2 y_2$$

for some constants c_1, c_2.

This is just because (by Lemma 1), $(Y_1 - Y_2)$ is a solution to (H), and as we saw in previous lectures any solution to (H) is a linear combination of y_1 and y_2.

Consider again (NH), and suppose we have

- y_1, y_2 - fundamental set of solutions to (H);
- Y_p - ANY one particular solution to (NH).

Then, if Y is any other solution to (NH),

$$Y - Y_p = c_1 y_1 + c_2 y_2 \text{ for some } c_1, c_2$$

$$Y = c_1 y_1 + c_2 y_2 + Y_p \text{ for some } c_1, c_2.$$

Let the nonhomogeneous 2^{nd} order equation:

$$(NH): \quad y'' + p(x)y' + q(x)y = g(x)$$

- Let $y_c = c_1 y_1 + c\, y_2$ be the general solution to its associated homogeneous eqn.:

$$(H): \quad y'' + p(x)y' + q(x)y = 0.$$

(y_c is called the complementary solution)

- Let y_p be any one particular solution to (NH).

(y_p is called the particular solution)

Then, the general solution to equation (NH) is:

$$\boxed{y = y_c + y_p = c_1 y_1 + c_2 y_2 + y_p}$$

Method of Undetermined Coefficients:

Setup: 2^{nd} order linear ODE w/ constant coefficients:

$$ay'' + by' + cy = g(x)$$

General Idea: make an assumption about the form of y_p with unknown coeff., and determine the coefficients.

Works with $g(x)$ of the forms: constant; polynomial; $e^{\alpha x}$; $\sin(\beta x)$; $\cos(\beta x)$; finite sums & products of these

① $y'' + 4y' - 2y = 2x^2 - 3x + 6$

- Solve homogeneous eqn.: $y'' + 4y' - 2y = 0$

$m^2 + 4m - 2 = 0$;

$\Delta = 16 + 8 = 24 \Rightarrow m = \dfrac{-4 \pm 2\sqrt{6}}{2} = -2 \pm \sqrt{6}$

$$y_c = c_1 e^{(-2-\sqrt{6})x} + c_2 e^{(-2+\sqrt{6})x}$$ (Complementary Sol.)

- Find particular solution:
  - Since $g(x) = 2x^2 - 3x + 6$ is a quadratic polynomial, assume
  $$y_p = Ax^2 + Bx + C$$
  - Find A, B, C such that y_p is a solution:

$$y_p' = 2Ax + B$$
$$y_p'' = 2A$$

$$2A + 4(2Ax + B) - 2(Ax^2 + Bx + C) = 2x^2 - 3x + 6$$

$$2A + 8Ax + 4B - 2Ax^2 - 2Bx - 2C = 2x^2 - 3x + 6$$

$$-2Ax^2 + (8A - 2B)x + (2A + 4B - 2C) = 2x^2 - 3x + 6$$

$$\begin{cases} -2A = 2 \\ 8A - 2B = -3 \\ 2A + 4B - 2C = 6 \end{cases}$$

$A = -1$

$-8 - 2B = -3$

$-2B = 5$

$B = -5/2$

$-2 - 10 - 2C = 6$

$-2C = 18$

$C = -9$

$\Rightarrow$ $y_p = -x^2 - \frac{5}{2}x - 9$ (Particular Solution)

$\Rightarrow$ General Solution:

$$y = C_1 e^{(-2-\sqrt{6})x} + C_2 e^{(-2+\sqrt{6})x} - x^2 - \frac{5}{2}x - 9$$

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(2) $y'' - y' + y = 2\sin(3x)$

- First guess for y_p: $y_p = A\sin(3x)$?

BUT: derivatives of $\sin(3x)$ will also generate $\cos(3x)$, so instead assume:

$$y_p = A\sin(3x) + B\cos(3x)$$

$$y_p' = 3A\cos(3x) - 3B\sin(3x)$$

$$y_p'' = -9A\sin(3x) - 9B\cos(3x)$$

$$\begin{aligned} y_p'' - y_p' + y_p &= -9A\sin(3x) - 9B\cos(3x) \\ &\quad - 3A\cos(3x) + 3B\sin(3x) \\ &\quad + A\sin(3x) + B\cos(3x) \\ &= \underbrace{(-8A+3B)}_{=2}\sin(3x) + \underbrace{(-8B-3A)}_{=0}\cos(3x) \end{aligned}$$

$$\begin{cases} -8A + 3B = 2 \\ -3A - 8B = 0 \end{cases} \Rightarrow B = -\frac{3}{8}A \Rightarrow -8A - \frac{9}{8}A = 2 \Rightarrow \frac{-73}{8}A = 2$$

$\Rightarrow$ $A = -\frac{16}{73}$; $B = \frac{6}{73}$

$$\boxed{y_p = \frac{-16}{73}\sin(3x) + \frac{6}{73}\cos(3x)}$$

Superposition Principle for Non-Hom. Linear ODE's:

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = g_1(x) \qquad y_{p_1}$$

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = g_2(x) \qquad y_{p_2}$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = g_n(x) \qquad y_{p_n}$$

$$\Rightarrow \quad a_2(x)y'' + a_1(x)y' + a_0(x)y = g_1(x) + g_2(x) + \ldots + g_n(x)$$

$$y_p = y_{p_1} + y_{p_2} + \ldots + y_{p_n}$$

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③ $y'' - 2y' - 3y = 4x - 5 + 6xe^{2x}$

$$\underbrace{y_{p_1} = Ax + B \qquad y_{p_2} = Cxe^{2x} + De^{2x}}_{\text{can do separately, but also works together}}$$

Homogeneous Sol.: $m^2 - 2m - 3 = 0$

$(m-3)(m+1) = 0$

$m_{1,2} = 3, -1$

$$y_c = C_1 e^{3x} + C_2 e^{-x}$$

$$y_p = Ax + B + Cxe^{2x} + De^{2x}$$

$$y_p' = A + Ce^{2x} + 2Cxe^{2x} + 2De^{2x}$$

$$= A + (C + 2D)e^{2x} + 2Cxe^{2x}$$

$$y_p'' = 2(C + 2D)e^{2x} + 2Ce^{2x} + 4Cxe^{2x}$$

$$= (4C + 4D)e^{2x} + 4Cxe^{2x}$$

$$y_p'' - 2y_p' - 3y_p = (4C+4D)e^{2x} + \cancel{4Cxe^{2x}}$$
$$- 2A - (2C+4D)e^{2x} - \cancel{4Cxe^{2x}}$$
$$- 3Ax - 3B - 3Cxe^{2x} - 3De^{2x}$$

match coefficients w/ $4x - 5 + 6xe^{2x}$

$$= \underbrace{(4C+4D-2C-4D-3D)}_{=0} e^{2x}$$
$$\underbrace{-3C}_{=6} xe^{2x} \rightarrow C = -2$$
$$\underbrace{-3A}_{=4} x \rightarrow A = -4/3$$
$$\underbrace{-2A-3B}_{=-5} \rightarrow \frac{8}{3} - 3B = -5$$

$$-3B = -5 - \frac{8}{3} = \frac{-23}{3}$$

$$B = \frac{23}{9}$$

$$2C - 3D = 0$$
$$D = \frac{2}{3}C \Rightarrow D = -4/3$$

$$\boxed{y_p = -\frac{4}{3}x + \frac{23}{9} - 2xe^{2x} - \frac{4}{3}e^{2x}}$$

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4 $y'' - 5y' + 4y = 8e^x$ → Reasonable assumption for y_p: Ae^x
(b/c e^x produces no new functions in its derivatives).

Homogeneous Solution:

$$m^2 - 5m + 4 = 0$$
$$(m-1)(m-4) = 0$$
$$m_{1,2} = 1, 4$$

$$y_c = c_1 e^x + c_2 e^{4x}$$

Glitch!

Since e^x is already a solution to the homogeneous eqn., it is not a choice for the particular solution of the non-homogeneous one.

$\Rightarrow$ New guess: $\boxed{y_p = Axe^x}$

$$y_p' = Ae^x + Axe^x$$

$$y_p'' = Ae^x + Ae^x + Axe^x = 2Ae^x + Axe^x$$

$$y_p'' - 5y_p' + 4y_p = 2Ae^x + \cancel{Axe^x} - 5Ae^x - \cancel{5Axe^x} + \cancel{4Axe^x}$$

$$= -3Ae^x = 8e^x$$

$$-3A = 8 \Rightarrow A = -8/3$$

$$y_p = -\frac{8}{3}xe^x$$

General Sol.: $\boxed{y = c_1e^x + c_2e^{4x} - \frac{8}{3}xe^x}$

Case 1: No function in the assumed particular solution is a solution to the homogeneous equation.

$g(x)$	Form of y_p
3 (constant)	A
$2x+1$	$Ax+B$
$6x^2-20$	Ax^2+Bx+C
x^3-2x+3	Ax^3+Bx^2+Cx+D
$\sin(8x)$	$A\sin(8x)+B\cos(8x)$
$\cos(2x)$	$A\sin(2x)+B\cos(2x)$
e^{9x}	Ae^{9x}
$(2x-4)e^{6x}$	$(Ax+B)e^{6x}$
x^2e^{3x}	$(Ax^2+Bx+C)e^{3x}$
$e^{2x}\sin(3x)$	$Ae^{2x}\sin(3x)+Be^{2x}\cos(3x)$
$x^2\sin(4x)$	$(Ax^2+Bx+C)\sin(4x)+(Dx^2+Ex+F)\sin(4x)$
$xe^{3x}\cos(4x)$	$(Ax+B)e^{3x}\cos(4x)+(Cx+D)e^{3x}\sin(4x)$

y_p = linear combination of all the linearly independent functions that are generated by repeated differentiation of $g(x)$

Case 2: A function in the assumed particular solution is also a solution of the associated homogeneous ODE.

If y_p contains terms that duplicate terms in y_c, then multiply by x^n, where n is the smallest power that eliminates duplication
(n can only be as high as 2 for second order equations).

Ex): $y'' - 2y' + y = e^x$

$y_c = c_1 e^x + c_2 x e^x \longrightarrow y_p = Ax^2 e^x$

Ex): $y'' - 6y' + 9y = 6x^2 + 2 - 12e^{3x}$

$y_c = c_1 e^{3x} + c_2 x e^{3x} \longrightarrow y_p = Ax^2 + Bx + C + Dx^2 e^{3x}$

Ex): $y'' + y = 2x\cos x$

$y_c = c_1 \sin x + c_2 \cos x \longrightarrow y_p = (Ax^2 + Bx)\sin x + (Cx^2 + Dx)\cos x$

Ex): $y'' + y = 2x\cos x + e^x \rightarrow y_p = (Ax^2 + Bx)\sin x + (Cx^2 + Dx)\cos x + Ee^x$

Ex): $y'' - 4y' + 4y = x + 3e^{2x}$

$y_c = c_1 e^{2x} + c_2 x e^{2x} \longrightarrow y_p = Ax + B + Cx^2 e^{2x}$

Ex): $y'' - 4y' + 4y = x^2 + 1 + \cos x \rightarrow y_p = Ax^2 + Bx + C + D\cos x + E\sin x$