Undetermined Coefficients – Part II.

Last time: non-homogeneous linear ODES:

$$y'' + p(x)y' + q(x)y = g(x)$$

The general solution is

$$y = y_c + y_p$$

where y_c is the complementary sol. (to the associated homogeneous eqn.)
and y_p is a particular solution to the non-homogeneous eqn.

Focus on second order equations w/ constant coefficients:

$$ay'' + by' + cy = g(x)$$

When $g(x)$ has certain forms:

- polynomial (or constant)
- $e^{\alpha x}$
- $\sin(\beta x)$ or $\cos(\beta x)$
- Sums & products of these

we can use the method of undetermined coefficients to find y_p:

- Main idea: make an assumption about the form of y_p (based on $g(x)$) with undetermined coefficients.

- y_p = linear combination of all distinct (linearly indp.) functions generated by $g(x)$, $g'(x)$ & $g''(x)$.

Examples from last time:

1) $y'' + 4y' - 2y = 2x^2 - 3x + 6$

$$y_c = c_1 e^{(-2-\sqrt{6})x} + c_2 e^{(-2+\sqrt{6})x}$$

$$y_p = Ax^2 + Bx + C \quad \text{(polynomial of same degree as } g\text{)}$$

$$\Rightarrow y_p = -x^2 - \frac{5}{2}x - 9$$

2) $y'' - y' + y = 2\sin(3x)$

$$y_c = c_1 e^{x/2} \sin\left(\frac{\sqrt{3}}{2}x\right) + c_2 e^{x/2} \cos\left(\frac{\sqrt{3}}{2}x\right)$$

$$y_p = A\sin(3x) + B\cos(3x)$$

$$y_p = -\frac{16}{73}\sin(3x) + \frac{6}{73}\cos(3x)$$

3) $y'' - 2y' - 3y = \underbrace{4x-5}_{Ax+B} + \underbrace{6x}_{(Cx+D)}e^{2x}$ Superposition

$$y_p = (Ax+B) + (Cx+D)e^{2x}$$

$$y_c = c_1 e^{3x} + c_2 e^{-x}$$

4) $y'' - 5y' + 4y = 8e^x$

$$y_c = c_1 e^x + c_2 e^{4x}$$

Case 1: No function in the assumed y_p duplicates any part of y_c

$g(x)$	y_p
Polynomial of degree $K \geq 0$	General Polynomial of degree K
$g(x) = x+1$	$y_p = Ax+B$
$g(x) = x^3+x-4$	$y_p = Ax^3+Bx^2+Cx+D$
$g(x) = 2$	$y_p = A$
Exponential $e^{\alpha x}$	$Ae^{\alpha x}$
$g(x) = 2e^{9x}$	$y_p = Ae^{9x}$
$\sin(\beta x)$ or $\cos(\beta x)$	$A\sin(\beta x) + B\cos(\beta x)$
$g(x) = 3\sin(2x)$	$y_p = A\sin(2x) + B\cos(2x)$
$g(x) = \cos(9x)$	$y_p = A\sin(9x) + B\cos(9x)$
(deg. K polynomial) $e^{\alpha x}$	(general K degree polynomial) $e^{\alpha x}$
$g(x) = (x^2+1)e^{3x}$	$y_p = (Ax^2+Bx+C)e^{3x}$
(deg. K poly.) $\sin(\beta x)$ or (deg. K poly.) $\cos(\beta x)$	(general deg. K poly.) $\sin(\beta x)$ + + (general deg. K poly.) $\cos(\beta x)$
$g(x) = (x-1)\sin(2x)$	$y_p = (Ax+B)\sin(2x) + (Cx+D)\cos(2x)$
$g(x) = x^2\cos(3x)$	$y_p = (Ax^2+Bx+C)\sin(3x) + (Dx^2+Ex+F)\cos(3x)$
$e^{\alpha x}\sin(\beta x)$ or $e^{\alpha x}\cos(\beta x)$	$Ae^{\alpha x}\sin(\beta x) + Be^{\alpha x}\cos(\beta x)$
$g(x) = e^{2x}\sin(7x)$	$y_p = Ae^{2x}\sin(7x) + Be^{2x}\cos(7x)$.
(deg. K poly) $e^{\alpha x}\sin(\beta x)$ or (deg. K poly) $e^{\alpha x}\cos(\beta x)$	(general deg. K poly.) $e^{\alpha x}\sin(\beta x)$ + + (general deg. K poly.) $e^{\alpha x}\cos(\beta x)$
$g(x) = (x+1)e^{2x}\cos(3x)$	$y_p = (Ax+B)e^{2x}\cos(3x) + (Cx+D)e^{2x}\sin(3x)$.

Case 2: Suppose $g(x) = g_1(x) + \ldots + g_k(x)$, where each function $g_j(x)$ is of the type previously discussed, with corresponding y_{p_j} for its particular solution. If any y_{p_j} contains a function in y_c, then multiply this y_{p_j} by x (possibly twice for 2nd order)

Example: $\boxed{y'' - 2y' + y = g(x)}$

Char. Eqn.: $m^2 - 2m + 1 = 0$; $(m-1)^2 = 0$; $m_1 = m_2 = 1$ (repeated root)

$$\boxed{y_c = c_1 e^x + c_2 x e^x}$$

$g(x) = 3e^{2x}$ $\Rightarrow$ $\boxed{y_p = Ae^{2x}}$ (no duplication)

$g(x) = 3x^2 e^{3x}$

$\boxed{y_p = (Ax^2 + Bx + C)e^{3x}}$ (no duplication)

$g(x) = 3x^2 e^x$

$$y_p \overset{?}{=} (Ax^2 + \underline{B}x + \underline{C})e^x = Ax^2 e^x + \underline{Bxe^x} + \underline{Ce^x}$$

both duplicate y_c

Multiply by x: is that enough?

$$Ax^3 e^x + Bx^2 e^x + \underline{\underline{Cxe^x}}$$

Still duplicates part of y_c

$\Rightarrow$ Multiply by x^2: $\boxed{y_p = (Ax^2 + Bx + C)x^2 e^x}$

(no more duplication)

(*) Look at what happens if we only multiplied by one x, and took $y_p = Ax^3e^x + Bx^2e^x + Cxe^x = (Ax^3 + Bx^2 + Cx)e^x$

Plug back into $y'' - 2y' + y = 3x^2e^x$:

$$y_p' = (3Ax^2 + 2Bx + C)e^x + (Ax^3 + Bx^2 + Cx)e^x$$
$$= \left(Ax^3 + (3A+B)x^2 + (2B+C)x + C\right)e^x$$

$$y_p'' = \left(3Ax^2 + (6A+2B)x + (2B+C)\right)e^x$$
$$+ \left(Ax^3 + (3A+B)x^2 + (2B+C)x + C\right)e^x$$
$$= \left(Ax^3 + (6A+B)x^2 + (6A+4B+C)x + (2B+2C)\right)e^x$$

$$y_p'' - 2y_p' + y_p = \Big(\left(\cancel{A - 2A + A}\right)x^3 + \left(\cancel{6A} + \cancel{B} - \cancel{6A} - \cancel{2B} + \cancel{B}\right)x^2$$
$$+ \left(6A + \cancel{4B} + \cancel{C} - \cancel{4B} - \cancel{2C} + \cancel{C}\right)x + \left(2B + \cancel{2C} - \cancel{2C}\right)\Big)e^x$$
$$= (6Ax + 2B)e^x$$
$$= 3x^2e^x$$

$6Ax + 2B$ cannot equal $3x^2e^x$!

And if we try $y_p = (Ax^4 + Bx^3 + Cx^2)e^x$? (after multiplying again by x)

$$y_p' = (4Ax^3 + 3Bx^2 + 2Cx + Ax^4 + Bx^3 + Cx^2)e^x$$
$$= \left(Ax^4 + (4A+B)x^3 + (3B+C)x^2 + 2Cx\right)e^x$$

$$y_p'' = \big(4Ax^3 + (12A+3B)x^2 + (6B+2C)x + 2C +$$
$$+ Ax^4 + (4A+B)x^3 + (3B+C)x^2 + 2Cx\big)e^x$$
$$= \left(Ax^4 + (8A+B)x^3 + (12A+6B+C)x^2 + (6B+4C)x + 2C\right)e^x$$

$$\Rightarrow y_p'' - 2y_p' + y_p = \Big[(A \cancel{-2A+A})x^4 + (\cancel{8A}+\cancel{B} - \cancel{8A} - 2B + B)x^3$$
$$+ (12A + \cancel{6B} + \cancel{C} - \cancel{6B} - 2C + C)x^2 +$$
$$+ (6B + 4C - 4C)x + 2C \Big] e^x$$

$$= \left[12A x^2 + 6Bx + 2C \right] e^x$$

$$= 3x^2 e^x$$

$$\begin{cases} 12A = 3 \\ 6B = 0 \\ 2C = 0 \end{cases} \qquad \begin{cases} A = 1/4 \\ B = 0 \\ C = 0 \end{cases} \qquad y_p = \frac{1}{4} x^4 e^x$$

Example: $y'' - y' - 2y = g(x)$, $\quad y_c = c_1 e^{2x} + c_2 e^{-x}$

$g(x) = e^{7x}$	$y_p = Ae^{7x}$ (no duplication)
$g(x) = 3e^{2x}$	$y_p = Axe^{2x}$ (duplicates e^{2x})
$g(x) = 2e^{-x} + \sin(8x)$	$y_p = Axe^{-x} + B\sin(8x) + C\cos(8x)$ duplication Superposition. Remark: $y_p = \left(Ae^{-x} + B\sin(8x) + C\cos(8x)\right)x$ is INCORRECT !
$g(x) = \sin(3x)e^{2x}$	$y_p = A\sin(3x)e^{2x} + B\cos(3x)e^{2x}$ (no duplication!)
$g(x) = x^2 e^{-x} + e^{ex}$	$y_p = (Ax^2 + Bx + C)xe^{-x} + De^{ex}$.