Integration by Parts:

$$\boxed{\int f(g)' = fg - \int (f)'g}$$

Comes from the Product Rule: $(fg)' = (f)'g + f(g)'$

$$\int (fg)' = \int (f)'g + \int f(g)'$$

$$\boxed{fg = \int (f)'g + \int f(g)'}$$

① $\int x e^{2x} dx$

→ want to differentiate x, b/c $(x)' = 1$
⟹ express the other function (e^{2x}) as a derivative:

$$\int x e^{2x} dx = \int x\left(\tfrac{1}{2}e^{2x}\right)' dx = (x)\cdot\left(\tfrac{1}{2}e^{2x}\right) - \int (x)'\left(\tfrac{1}{2}e^{2x}\right) dx$$

$$= \tfrac{1}{2}xe^{2x} - \tfrac{1}{2}\int e^{2x} dx$$

$$= \boxed{\tfrac{1}{2}xe^{2x} - \tfrac{1}{4}e^{2x} + C}$$

② $\int x \sin x \, dx$

$$= \int x(-\cos x)' dx$$

$$= -x\cos x + \int \cos x \, dx$$

$$= \boxed{-x\cos x + \sin x + C}$$

③ $\int x \ln x \, dx$

$$= \int \left(\frac{x^2}{2}\right)' \ln x \, dx$$

$$= \frac{x^2}{2}\ln x - \int \frac{x^2}{2}\cdot\frac{1}{x} dx$$

$$= \boxed{\frac{x^2}{2}\ln x - \frac{x^2}{4} + C}$$

(4) $\int x^2 e^{-x}\,dx$

$= \int x^2(-e^{-x})'\,dx$

$= -x^2e^{-x} + \int 2xe^{-x}\,dx$

by parts again

$= -x^2e^{-x} - 2\int x(e^{-x})'\,dx$

$= -x^2e^{-x} - 2xe^{-x} + 2\int e^{-x}\,dx$

$= \boxed{-x^2e^{-x} - 2xe^{-x} - 2e^{-x} + C}$

(5) $\int \arctan(x)\,dx$

$= \int (x)'\arctan(x)\,dx$

$= x\arctan(x) - \int x\cdot\frac{1}{1+x^2}\,dx$

$= \boxed{x\arctan(x) - \frac{1}{2}\ln(1+x^2) + C}$

(6) $\int e^x \sin(x)\,dx$

$I := \int e^x\sin(x)\,dx$

$= \int (e^x)'\sin(x)\,dx = e^x\sin(x) - \int e^x\cos(x)\,dx$

by parts again

$= e^x\sin(x) - \int (e^x)'\cos(x)\,dx$

$= e^x\sin(x) - e^x\cos(x) - \underbrace{\int e^x\sin(x)\,dx}_{I}$

$\Rightarrow 2I = e^x\sin(x) - e^x\cos(x)$

$\Rightarrow \boxed{I = \frac{1}{2}e^x(\sin(x) - \cos(x))}$

⑦ $\displaystyle\int \frac{(\ln x)^2}{x^3}\,dx$

<u>Solution I</u> (Integration by Parts):

$$\int \frac{(\ln x)^2}{x^3}dx = \int (\ln x)^2 \cdot \left(\frac{-1}{2x^2}\right)' dx$$

$$= -\frac{1}{2x^2}(\ln x)^2 + \int \frac{1}{2x^2}\cdot 2(\ln x)\cdot\frac{1}{x}$$

$$= -\frac{1}{2x^2}(\ln x)^2 + \int \frac{1}{x^3}(\ln x)$$

$$= -\frac{1}{2x^2}(\ln x)^2 + \int \left(\frac{-1}{2x^2}\right)'(\ln x)\,dx$$

$$= -\frac{1}{2x^2}(\ln x)^2 - \frac{1}{2x^2}(\ln x) + \int \frac{1}{2x^2}\cdot\frac{1}{x}dx$$

$$= \boxed{-\frac{1}{2x^2}(\ln x)^2 - \frac{1}{2x^2}(\ln x) - \frac{1}{4x^2} + C}$$

<u>Solution II</u>: (Substitution + by Parts)

$$\left.\begin{aligned} u &= \ln x \Rightarrow x = e^u \\ du &= \frac{1}{x}dx \Rightarrow dx = x\,du = e^u du \end{aligned}\right\} \Rightarrow \int \frac{u^2}{(e^u)^3}e^u du = \int u^2 e^{-2u} du$$

$$\int u^2 e^{-2u}du = -\frac{1}{2}u^2e^{-2u} + \int u e^{-2u}du$$

$$= -\frac{1}{2}u^2e^{-2u} - \frac{1}{2}ue^{-2u} - \frac{1}{4}e^{-2u} + C$$

$$\Rightarrow \boxed{-\frac{1}{2}(\ln x)^2(x^{-2}) - \frac{1}{2}(\ln x)(x^{-2}) - \frac{1}{4}(x^{-2}) + C}$$

⑧ $\int \frac{1}{x-2}\,dx = \boxed{\ln|x-2| + C}$

⑨ $\int \frac{x^2-2x}{x^3-3x^2+1}\,dx = \boxed{\frac{1}{3}\ln|x^3-3x^2+1| + C}$

<u>Remark</u> : $(x^3-3x^2+1)' = 3x^2-6x = 3(x^2-2x)$

⑩ $\int \frac{s+2}{s-3}\,ds = \int \frac{(s-3)+5}{s-3}\,ds = \int\left(1+\frac{5}{s-3}\right)ds = \boxed{s+5\ln|s-3|+C}$

⑪ $\int \frac{x}{x^2+2x-3}\,dx = \int \frac{x}{(x-1)(x+3)}\,dx$

<u>Partial Fraction Decomposition</u> : $\frac{x}{(x-1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+3}$

$x = Ax+3A+Bx-B$

$x = (A+B)x + (3A-B)$

$\begin{cases} A+B=1 \\ 3A-B=0 \Rightarrow B=3A \end{cases}$

$\Rightarrow 4A=1 \Rightarrow A=1/4;\ B=3/4$

$$\int \frac{x}{(x-1)(x+3)}\,dx = \frac{1}{4}\int\frac{1}{x-1}\,dx + \frac{3}{4}\int\frac{1}{x+3}\,dx$$

$$= \boxed{\frac{1}{4}\ln|x-1| + \frac{3}{4}\ln|x+3| + C}$$

⑫ $\int \frac{3+x^2}{x}\,dx = \int\left(\frac{3}{x}+x\right)dx = \boxed{3\ln|x| + \frac{x^2}{2} + C.}$

⑬ $\int \frac{1}{x(3+\ln x)}\,dx = \int \frac{1}{3+u}\,du \quad [u=\ln x]$

$= \ln|3+u| + C = \boxed{\ln|3+\ln x| + C}$

⑭ $\int \frac{x^2}{1+x^2}\,dx = \int\frac{(1+x^2)-1}{1+x^2}\,dx = \int\left(1-\frac{1}{1+x^2}\right)dx = \boxed{x - \arctan x + C}$